Journal of Civil Engineering and Urban Planning (2025)
Clausius Scientific Press, Canada
DOI: 10.23977/jceup.2025.070116
ISSN 2616-3969 Vol. 7 Num. 1

Application of Copula Entropy in Constructing Two-Dimensional Joint PDF

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Keywords: Two-Dimensional Joint PDF; Copula Theory; Copula Entropy

Abstract: Two-dimensional joint distribution functions are extensively employed across multiple disciplines, serving as a critical tool to describe and quantify the intricate relationships between two random variables. Copula theory is widely used in the calculation of joint distribution function due to its flexibility in selection of marginal distributions and the dependency structure However, existing methods of constructing joint distributions based on Copula theory often face issues regarding the selection of an appropriate Copula function, which may be inefficient and inflexible. To address these issues, this paper proposes a method based on copula entropy. By using the maximum entropy copula entropy theory as the creteria, a two-dimensional joint probability density function is proposed. Case studies demonstrate that the copula entropy method is not limited to existing copula types and exhibits superior computational efficiency and greater flexibility compared to conventional copula methods when handling mixed distributions and multimodal distributions.

1. Introduction

Constructing joint probability density function (PDF) between variables is crucial in engineering and structural reliability, as engineering systems and structures are typically influenced by multiple random variables, such as material strength, external loads, and environmental conditions. These variables may exhibit correlations, and such correlations have a significant impact on the system's performance and reliability. In the field of structural reliability need to consider the relationships between multiple random variables, typically required the estimation of joint PDF from limited sample data. This can be achieved through various statistical methods, such as histograms and kernel density estimation. However, these methods often face difficulties when dealing with high-dimensional data or complex dependency structures between variables, and neither method can provide a specific parametric expression. There are some existing two-dimensional distribution models, such as the two-dimensional Gaussian distribution, two-dimensional log-normal distribution, and two-dimensional exponential distribution, which can account for correlations between data. However, these models require the marginal distributions to have the same distribution type; for example, two-dimensional Gaussian distribution requires both marginal distributions to be normally distributed. In the field of structural reliability, the distribution types of variables are often uncertain, the existing two-dimensional distribution models cannot accurately represent the correlation structure between variables. To facilitate modeling of diverse random variables, Li etal. [1] proposed a flexible bivariate distribution model, and is derived with the probability equivalently expressed as the summation of three basic probabilities corresponding to simple functions. These three basic probabilities are calculated with the aid of univariate cubic normal distribution, and thus the proposed model is named as bivariate cubic normal (BCN)distribution. Wu etal. [2] studied two approximation methods for constructing joint PDFS under incomplete probabilistic information: the approximation method P based on the Pearson correlation coefficient and the approximation method S based on the Spearman correlation coefficient. They used the Nataf distribution model [3] to obtain the joint PDF between variables. However, the variables need to be transformed into normal distributions through an equiprobable transformation, and the correlation structure is described using linear coefficients, which cannot solve complex correlation structures. Additionally, there are many researchers have used the Monte Carlo method, based on the characteristics and assumptions of real-world data, to recreate certain processes through computer simulations. Hawkes, Gouldby, etal. [4] used a joint probability analysis method with Monte Carlo simulations to fit the distributions of water levels, wave heights, and wave steepness, as well as their correlations. Adamson etal. [5] used the Monte Carlo method to analyze river flooding issues by representing the correlations between variables through a series of conditional distributions. However, Monte Carlo method require a large number of sample simulations to estimate results, what is difficult to effectively handle complex correlation structures between some variables.

Sklar first proposed Sklar's Theorem [6] in 1959 and established Copula theory, which creates a bridge between joint distributions and marginal distributions. Nelson [7] pointed out that Copula theory provides a new approach for constructing joint distribution functions under incomplete probabilistic information. Copula function connects the joint distribution of variables with their marginal distributions and is essentially a joint distribution function. By differentiating the Copula function, we can obtain the joint PDF. At present, the commonly used Copula functions are mainly classified into elliptical Copulas and Archimedean Copulas. However, a single Copula function has some limitations in describing the correlation structure between variables. In practical engineering applications, the correlation structures between variables are often quite complex, if only use a single Copula function may have somewhat restrictive.Hu [8] linearly combined commonly used Archimedean Copula functions, such as the Frank Copula, Gumbel Copula, and Clayton Copula, to create a new mixed Copula model. The advantage of the mixed Copula is that it can capture more different Copulas characteristics, offering greater flexibility and better describing complex correlation structures compared to a single Copula. In recent years, Cook[9] proposed the OEN mixed model to solving the joint distribution of wind speed and wind direction. The marginal distributions of the variables were used a mixture distribution, and multiple Gaussian Copulas were used to represent the correlation structure between variables. JI [10] developed the generalized bivariate mixture (GBM) model for directional wind speed using two-dimensional Copula functions, which Copula functions used include normal, Frank, and AMH and the marginal distributions include normal and logistic. There are 12 sub-models used to fit the correlation structure between variables, offering greater flexibility and freedom. Constructing joint PDF based on Copula theory, whether using a single Copula function or mixed Copula functions, the primary challenge is selecting an appropriate Copula function to describe the correlation structure. Up to now, there is no universally accepted method for selecting Copula functions, and most approaches are to select a few commonly used Copula functions, Dias and Embrechts used the AIC [11] (Akaike Information Criterion) and BIC [12] (Bayesian Information Criterion) to select the most suitable Copula function. However, there are many types of Copula functions, each of them have distinct characteristics and can describe different correlation structures [13]. For example, the Independent Copula cannot reflect the impact of correlation between variables on the joint distribution function; the Gaussian and Frank Copulas have symmetric correlation structures; the Clayton and CClayton Copulas have lower tail and upper tail dependencies. The correlation structures between various variables are generally uncertain and diverse. If only select several existing Copula functions to compare to get a better Copula function, it can not really reflect the correlation structure between variables and the comparison efficiency between multiple Copula functions will be relatively low. In practical applications, selecting different Copula functions for the same problem can lead to different analytical results. Therefore, it is very necessary to select the Copula function which is suitable for the correlation structure between variables.

To solve disadvantages of inflexibility and inefficiency in constructing two-dimensional joint PDF based on Copula theory, this paper introduces a method for constructing two-dimensional joint PDF based on Copula entropy. Copula entropy combines the advantages of Copula theory and the concept of information entropy, which can provide a more flexible and accurate method for constructing joint PDF. No longer make artificial assumptions about Copula function types, this approach directly solves the two-dimensional Copula probability density function using the Copula entropy formula through the Lagrange multiplier method and genetic algorithm based on variable data , then combined with the marginal distributions of the variable data itself to obtain the two-dimensional joint PDF.

2. Constructing Two-Dimensional Joint PDF Based on Copula Theory

Copula theory can combine the marginal distributions of multiple random variables into a multivariate distribution function, effectively describing the correlations between multivariate variables without restricting the types of marginal distributions. Therefore, a two-dimensional Copula can be used to construct the joint PDF of two-dimensional random variables.

2.1 Sklar Theorem

Sklar theorem [6] is the core of Copula theory. It allows the joint distribution of multivariate random variables to be decomposed into the form of marginal distributions and a Copula function, and the Copula function describing the correlations between the variables. In fact a Copula function is a type of joint distribution function. For the case of two-dimensional random variables, a Copula function is defined as a two-dimensional joint distribution function with uniform marginal distributions [7] on the interval [0, 1] in the $[0, 1]^2$ space. According to Sklar theorem, the two-dimensional joint distribution function of random variables x_1 and x_2 is given by:

$$F(x_1, x_2) = C(F(x_1), G(x_2); \alpha) = C(u, v; \alpha) \tag{1}$$

where $u = F(x_1)$ and $v = G(x_2)$ are the marginal distribution functions of x_1 and x_2 respectively;C is the two-dimensional Copula function, and α represents the correlation parameters of the Copula function. From the relationship between the joint distribution function and the joint PDF, the two-dimensional joint PDF is given by:

$$f(x_1, x_2) = c(F(x_1), G(x_2); \alpha) f(x_1) g(x_2) \tag{2}$$

where c is the probability density function of the Copula, and $f(x_1)$ and $g(x_2)$ are the marginal probability density functions of x_1 and x_2 respectively.

2.2 Comparison of Copula Function

Copula functions can directly combine random variables following various distributions to form a multivariate distribution, effectively describing the dependencies between variables. However, different types of Copula functions capture these dependencies in different ways, each with a unique correlation structure. Commonly used Copula functions can be classified into elliptical Copulas and Archimedean Copulas. Elliptical Copulas include the Gaussian Copula and t-Copula, while Archimedean Copulas include the Gumbel Copula, Frank Copula, and Clayton Copula. The joint distribution functions, joint PDF, and the range of Copula parameters for these five two-dimensional Copula functions are listed in Table 1. For the case of two-dimensional random variables, the dependencies between the variables are not fixed. Therefore, to select an appropriate type of Copula function that describes the optimal dependency structure between variables, a goodness-of-fit test for Copula functions is necessary. The commonly used AIC [11] and BIC [12] are important tools for selecting the most suitable Copula function. The AIC criterion aims to select the best model by balancing model complexity (the number of parameters) and model fit (the likelihood function), generally favoring more complex models. The BIC criterion, based on the AIC criterion, tends to favor simpler models.

$$AIC = 2k - 2\ln(\hat{L}) \tag{3}$$

$$BIC = k\ln(n) - 2\ln(\hat{L}) \tag{4}$$

where k is the number of parameters; $\hat{L}$ is the maximum likelihood of the model; n is the sample size. The smaller the AIC and BIC values, the better the model fit. The root mean square error (RMSE) [14] is also commonly used to evaluate goodness-of-fit. Its calculation formula is as follows:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}[F_i(x,y)] - \bar{F}_i(x,y)]^2} \tag{5}$$

where $F_i(x, y)$ represents the theoretical or empirical value; $\bar{F}_i(x, y)$ denotes the modeled value; n is the sample size. A smaller RMSE value indicates better model fit. This study employs both AIC and RMSE as evaluation metrics for model goodness-of-fit.

Table 1 CDFs, PDFs, and Parameter Ranges for 5 Types of two-dimensional Copulas

Type of Copula	$C(u,v;\alpha)$	$c(u,v;\alpha)$	Parameter Range
Gaussian	$\Phi_\rho(\Phi^{-1}(u),\Phi^{-1}(v))$	$\frac{\phi_\rho(\Phi^{-1}(u),\Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$	$\alpha \in (-1,1)$
t	$T_v^{-1}(T_v(u),T_v(v))$	$\frac{t_v(T_v(u),T_v(v))}{t_v(u)t_v(v)}$	$\alpha \in (-1,1)$
Gumbel	$exp[-((-\ln u)^\alpha + (-\ln v)^\alpha)^{\frac{1}{\alpha}}]$	$\frac{uv(-\ln u)^{\alpha-1}(-\ln v)^{\alpha-1}}{[\alpha - 1 + (-\ln u)^\alpha + (-\ln v)^\alpha]^2}$	$\alpha \in (1,+\infty)$
Clayton	$[\max\{(u^{-\alpha} + v^{-\alpha} - 1), 0\}]^{-\frac{1}{\alpha}}$	$(u^{-\alpha} + v^{-\alpha} - 1)^{-\frac{1}{\alpha}-2}(u^{-\alpha}v^{-\alpha})(\alpha+1)$	$\alpha \in (-1,0)\cup(0,+\infty)$
Frank	$-\frac{1}{\alpha}ln[1+\frac{(e^{-\alpha u}-1)(e^{-\alpha v}-1)}{e^{-\alpha}-1}]$	$\frac{(e^{-\alpha}-1)e^{-\alpha(u+v)} + (e^{-\alpha u}-1)(e^{-\alpha v}-1)}{\alpha(e^{-\alpha}-1+e^{-\alpha u+\alpha v})^2}$	$\alpha \in (-\infty,0)\cup(0,+\infty)$

3. Constructing Two-Dimensional Joint PDF Based on Copula Entropy

3.1 Copula Entropy

Copula entropy is a method for measuring the complex dependencies among variables in a multivariate distribution. It combines the concept of entropy from information theory and the Copula function from statistics. Entropy measures the uncertainty of random variables, while the Copula function describes the dependence structure among multiple random variables. For two-dimensional random variables x_1 and x_2, the Copula entropy is given by:

$$H(x_1, x_2) = -\int_0^1 \int_0^1 c(u,v) \log c(u,v) dudv \tag{6}$$

where $u = F(x_1)$ and $v = F(x_2)$ are the marginal distribution functions of the two random variables x_1 and x_2 respectively, c is the two-dimensional Copula probability density function, and H is the Copula entropy. According to the principle of maximum entropy [15], when H reaches its maximum value, the corresponding $c(u,v)$ is the optimal function describing the relationship between the two random variables x_1 and x_2, the optimal Copula probability density function describing the relationship between the two variables. Based on this, the two-dimensional joint PDF can be obtained as follows:

$$f(x_1, x_2) = c(F(x_1), G(x_2)) f(x_1) g(x_2) \tag{7}$$

where $f(x_1)$ and $g(x_2)$ are the marginal probability density functions of the random variables x_1 and x_2 respectively, and $c(u,v)$ is the two-dimensional Copula probability density function derived based on Copula entropy.

3.2 Solving the two-dimensional Copula Probability Density Function

From Eq. 5, each entropy value corresponds to a two-dimensional Copula probability density function. According to the principle of maximum entropy, the $c(u,v)$ corresponding to the maximum H is the optimal two-dimensional Copula probability density function describing the relationship between the two variables. By using the Lagrange multiplier method, we maximize the entropy H under the given constraint conditions [16] of the Copula probability density function to solve for $c(u,v)$. The constraint conditions are as follows:

$$\int_0^1 \int_0^1 c(u,v) dudv = 1 \tag{8}$$

$$\int_0^1 \int_0^1 u^r c(u,v) dudv = \int_0^1 \int_0^1 v^r c(u,v) dudv = \frac{1}{r+1} \tag{9}$$

$$\int_0^1 \int_0^1 uvc(u,v) dudv = \frac{\rho+3}{12} \tag{10}$$

where $r = 0,1,2 \ldots$; ρ is the Spearman rank correlation coefficient. Using these constraints, the Lagrange equation is given by:

$$L_c = -\int_0^1\int_0^1 c(u,v)\ln[c(u,v)]dudv - (\lambda_0 - 1)[\int_0^1\int_0^1 c(u,v)dudv - 1] - \sum_{i=1}^{2}\lambda_i[\int_0^1\int_0^1 u^i c(u,v)dudv - \frac{1}{i+1}]$$
$$-\sum_{i=1}^{2}\lambda_i[\int_0^1\int_0^1 v^i c(u,v)dudv - \frac{1}{i+1}] - \lambda_3[\int_0^1\int_0^1 uvc(u,v)dudv - \frac{\rho+3}{12}] \quad (11)$$

where $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are the unknown Lagrange multipliers. By solving the Lagrange equation, the Copula probability density function $c(u, v)$ [16] can be obtained:

$$c(u,v) = \exp(-\lambda_0 - \sum_{i=1}^{2}\lambda_i u^i - \sum_{i=1}^{2}\lambda_i v^i - \lambda_3 uv) \quad (12)$$

where λ_0 can be expressed in terms of λ_1, λ_2 and λ_3:

$$\lambda_0 = \ln\{\int_0^1\int_0^1 \exp[-\sum_{i=1}^{2}(\lambda_i u^i + \lambda_i v^i) - \lambda_3 uv]dudv\} \quad (13)$$

Combining with Eq. 11, solving for the Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3$ yields the two-dimensional Copula probability density function $c(u, v)$. Mead [17] and Kapur [18] proposed that the model parameters can be estimated by minimizing a convex function:

$$\Gamma = \lambda_0 + \sum_{i=1}^{2}(\lambda_i u^i + \lambda_i v^i) + \lambda_3 uv \quad (14)$$

Eq. 13 can be solved using a genetic algorithm to obtain the global optimal solutions $\lambda_1, \lambda_2, \lambda_3$. From Eq. 11, the two-dimensional Copula probability density function is:

$$c(u,v) = \frac{\exp(-\lambda_1 u - \lambda_2 u^2 - \lambda_1 v - \lambda_2 v^2 - \lambda_3 uv)}{\int_0^1\int_0^1 \exp(-\lambda_1 u - \lambda_2 u^2 - \lambda_1 v - \lambda_2 v^2 - \lambda_3 uv)dudv} \quad (15)$$

4. Numerical Examples

4.1 Example 1: Data with Two Different Distributions

A sample of 1000 randomly generated data points was created. The random variable x follows a Gamma distribution with location parameter $\mu_1 = 2$ and scale parameter $\beta_1 = 2$, while random variable y follows a Gumbel distribution with location parameter $\mu_2 = 2$ and scale parameter $\beta_2 = 2$. The marginal distributions of both variables are shown in the following figure 1 and 2:

(1) Five bivariate copula functions were selected from Table 1 to construct the joint distribution function of random variables x and y, expressed as Equation (1). The corresponding joint probability density function was then derived by differentiating these copula functions:

$$f(x, y) = c(F(x), G(y); \alpha) f(x) g(y) \quad (16)$$

where $c(u, v) = \frac{\partial C^2(u,v)}{\partial u\, \partial v}$; $f(x)$ and $g(y)$ are the marginal probability density functions of variables x and y respectively.

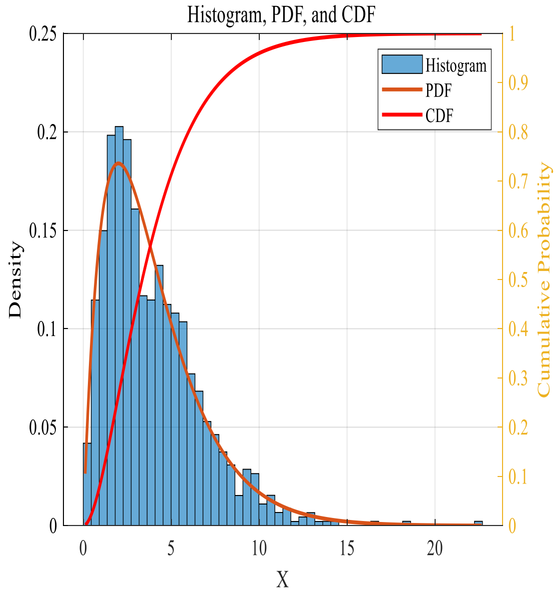

Fig.1 Histogram, PDF and CDF of x

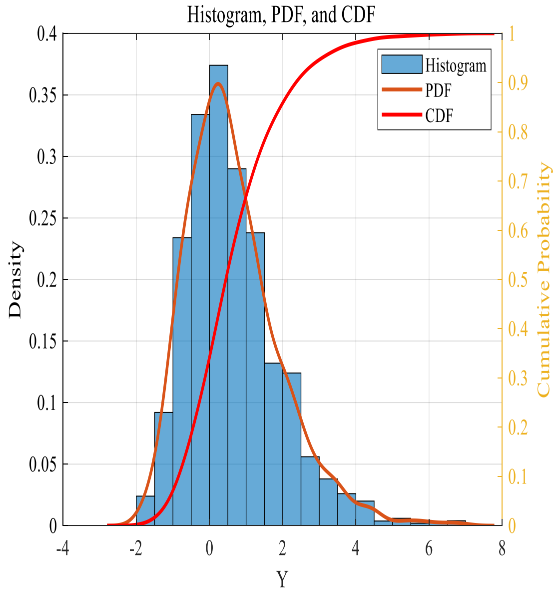

Fig.2 Histogram, PDF and CDF of y

(2) Using the sample data points in Figures 1 and 2, We estimate the parameters of the connection function by maximum likelihood, as shown in Table 2.

Table 2. Parameter Values α, AIC, RMSE

	Gaussian	t	Gumbel	Clayton	Frank
α	[1,0.0103;0.0103,1]	[1,0.011;0.011,1]	1.0155	0.00145	0.1029
AIC	7661	7657	7657	7659	7659
RMSE	0.0039	0.0039	0.004	0.0039	0.004

(3) Employing the same 1000 sets of random sample points, we construct the two-dimensional joint probability density function (PDF) through Copula entropy. Subsequently, the Lagrange multipliers λ_i are determined by implementing the methodology elaborated in Section 3.2, with computational results systematically organized in Table 3.

Table 3. Lagrange Multipliers λ_i

λ_i	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
Parameter	-0.058	0.116	0.002	0.116	0.002	-0.236

From Tab. 3, the two-dimensional Copula probability density function is:

$$c(u,v)=\frac{\exp(-0.116u-0.002u^2-0.116v-0.002v^2+0.2361uv)}{\int_0^1\int_0^1 \exp(-0.116u-0.002u^2-0.116v-0.002v^2+0.2361uv)dudv} \tag{17}$$

(4) Building upon both the conventional copula method and the copula entropy method to obtain the bivariate copula density functions, we then derived the joint probability density function of variables X and Y using Equation (16). The results are presented in the figure 3 to 9 below:

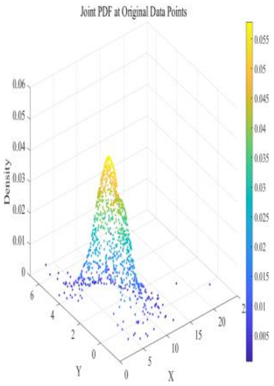

Fig.3 Original Data Joint PDF

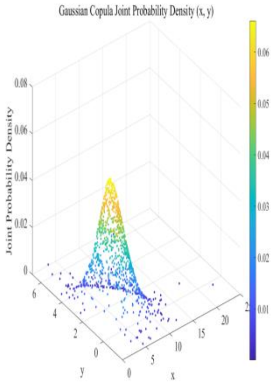

Fig.4 Gaussian Copula Joint PDF

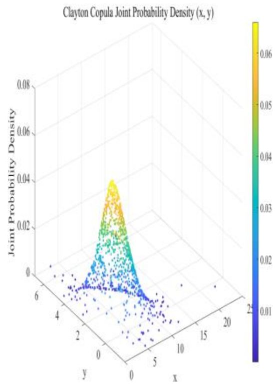

Fig.5 Clayton Copula Joint PDF

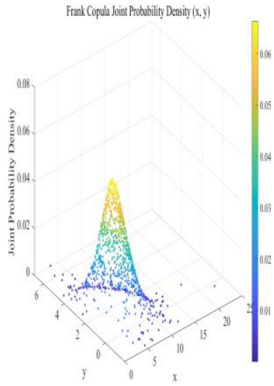

Fig.6 Frank Copula Joint PDF

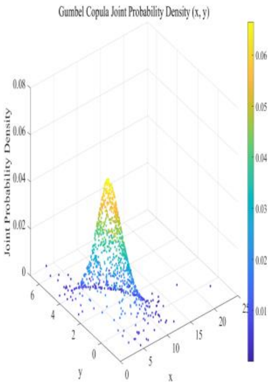

Fig.7 Gumbel Copula Joint PDF

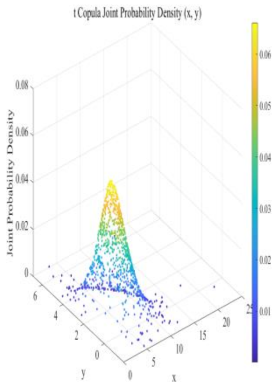

Fig.8 t Copula Joint PDF

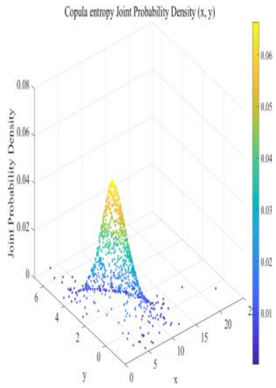

Fig.9 Copula entropy Joint PDF

Comparison of Results

The model-derived joint probability density functions obtained from both conventional copula and copula entropy methods were compared against theoretical values. The goodness-of-fit was evaluated using AIC and RMSE metrics, with comparative results presented in the following table 4:

Table 4. Comparison of Results

Methods	Type of Copula	AIC	RMSE
Copula	Gaussian	7661	0.0039
	t	7663	0.0039
	Gumbel	7657	0.004
	Clayton	7659	0.0039
	Frank	7659	0.004
Copula entropy	-	7664	0.004

As evidenced in the table, the copula entropy method yields AIC and RMSE values of 7664 and 0.004 respectively, demonstrating comparable performance to conventional elliptical and Archimedean copulas. These results confirm that the copula entropy approach achieves satisfactory fitting performance for mixed-distribution data and accurately characterizes the dependence structure between variables.

4.2 Example 2: Multimodal Distribution Data

Consider two datasets of wind speed and wind direction, where the marginal distribution of wind direction exhibits multimodal characteristics. Here, we treat wind speed and wind direction as random variables x and y, respectively. The marginal distributions of both variables are shown in the figure 10 and 11 below:

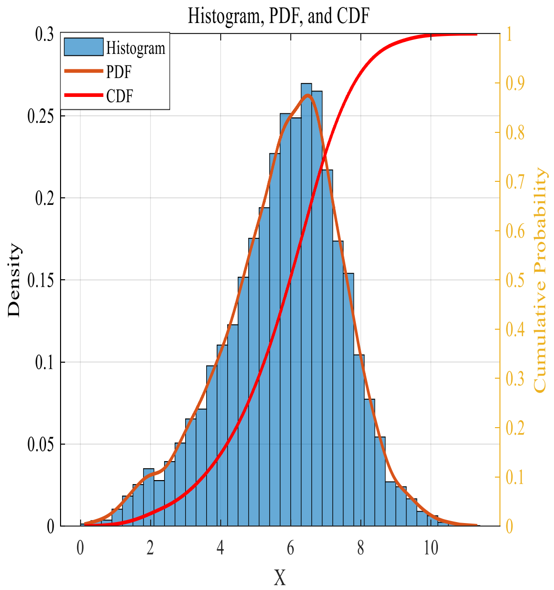

Fig.10 Histogram, PDF and CDF of X

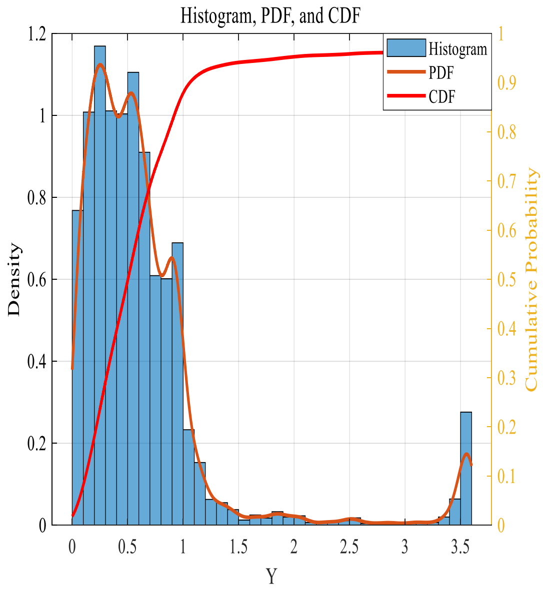

Fig.11 Histogram, PDF and CDF of Y

(1) Five bivariate copula functions were selected from Table 1 to construct the joint distribution function of random variables x and y, expressed as Equation (1). The corresponding joint probability density function was then derived by differentiating these copula functions:

$$f(x,y) = c(F(x),G(y);\alpha)\, f(x)g(y) \tag{18}$$

Where $c(u,v) = \frac{\partial C^2(u,v)}{\partial u\, \partial v}$; $f(x)$ and $g(y)$ are the marginal probability density functions of variables x and y respectively.

(2) Using the sample data points, we estimated the parameters of the copula functions via maximum likelihood estimation and computed both AIC and RMSE values, as shown in Table 5.

Table 5. Parameter Values α, AIC, RMSE

	Gaussian	t	Gumbel	Clayton	Frank
α	[1,-0.0114;-0.0114,1]	[1,0.0237;0.0237,1]	1.0095	0.00145	0.2391
AIC	45761	45725	45758	45760	45755
RMSE	0.0278	0.0265	0.0268	0.0274	0.0260

(3) Employing the same 1000 sets of random sample points, we construct the two-dimensional joint probability density function (PDF) through Copula entropy. Subsequently, the Lagrange multipliers λ_i are determined by implementing the methodology elaborated in Section 3.2, with computational results systematically organized in Table 6.

Table 6. Lagrange Multipliers λ_i

λ_i	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5
Parameter	-0.018	0.231	0.01	0.231	0.01	-0.481

From Table 6, the two-dimensional Copula probability density function is:

$$c(u,v)=\frac{\exp(-0.231u-0.01u^2-0.231v-0.01v^2+0.481uv)}{\int_0^1\int_0^1 \exp(-0.231u-0.01u^2-0.231v-0.01v^2+0.481uv)dudv} \tag{19}$$

(4) Building upon both the conventional copula method and the copula entropy method to obtain the bivariate copula density functions, we then derived the joint probability density function of variables X and Y using Equation (18). The results are presented in the figure 12 to 18 below:

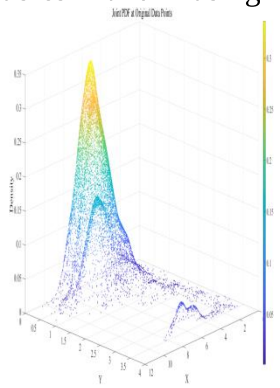

Fig.12 Original Data Joint PDF

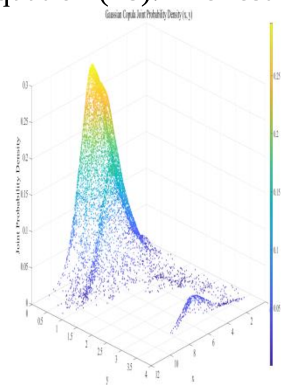

Fig.13 Gaussian Copula Joint PDF

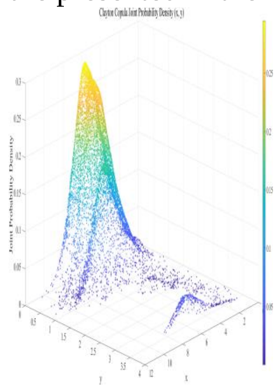

Fig.14 Clayton Copula Joint PDF

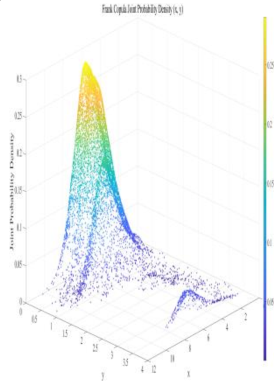

Fig.15 Frank Copula Joint PDF

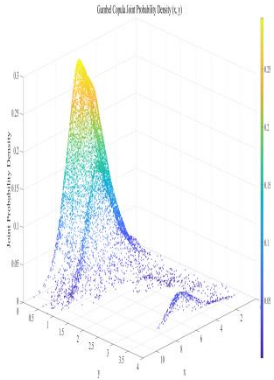

Fig.16 Gumbel Copula Joint PDF

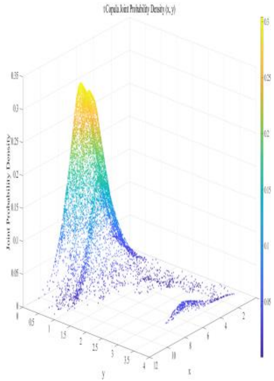

Fig.17 t Copula Joint PDF

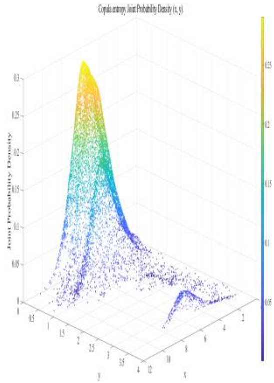

Fig.18 Copula entropy Joint PDF

Comparison of Results

The model-derived joint probability density functions obtained from both conventional copula and copula entropy methods were compared against theoretical values. The goodness-of-fit was evaluated using AIC and RMSE metrics, with comparative results presented in the following table 7:

Table 7. Comparison of Results

Methods	Type of Copula	AIC	RMSE
Copula	Gaussian	45761	0.0278
	t	45765	0.0264
	Gumbel	45758	0.0267
	Clayton	45760	0.0274
	Frank	45755	0.0260
Copula entropy	-	45751	0.0259

The results in Table 7 indicate that the copula entropy method achieves superior performance with AIC = 45,751 and RMSE = 0.0259, which are significantly lower than those obtained from conventional elliptical and Archimedean copulas. This demonstrates that the copula entropy approach provides better fitting performance for multimodal distribution data and more accurately characterizes the dependence structure between variables.

5. Conclusion

The copula entropy method combines the maximum entropy principle with copula theory, retaining copula theory's advantage of constructing multivariate distributions from random variables with different marginal distributions to describe their dependencies, while overcoming the limitations of traditional copula theory. Unlike being constrained to existing copula function types, this method directly computes the relationships between variables from data, making the function types more flexible.

Moreover, it eliminates the need to assume copula function types and reduces redundant computations, thereby improving computational efficiency to some extent. Since it performs calculations directly using variable data, even for complex mixed distributions, the copula entropy method can solve the two-dimensional copula probability density function and consequently obtain a concise and easily expressible two-dimensional joint probability density function.

However, the copula entropy method has poorer accuracy and unsatisfactory fitting performance when dealing with two-dimensional joint probability density functions between random variables with identical distributions. The fundamental issue lies in the fact that the coefficients obtained through Newton's iteration method are not optimal. Therefore, how to obtain the optimal coefficients requires further in-depth research.

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