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# Cauchy Convergence Criterion in the Split Complex Plane

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***Abstract:*** This paper investigates the Cauchy convergence criterion in the split complex plane. The split complex numbers are generated by two real numbers and form a commutative ring with zero divisors. Since split complex numbers have zero divisors, this paper concludes the Cauchy convergence criterion in the split complex plane according to the decomposition property of split complex numbers. This result can not only lay the foundation for the development of the split complex analysis but also can infuse research momentum into the application of the split complex numbers in terms of physics.

## 1. Introduction

The split complex numbers have experienced considerable progress in recent years. The space of bicomplex numbers has been illustrated as the first in an infinite sequence of multicomplex spaces and the generalizations of the space of complex numbers [1]. Academic researchers have study some properties of the elements of the space of split complex numbers, the trace, the determinant, invertibility conditions, and some elementary functions and contrasted with the usual complex functions [2]. The hyperbolic numbers were first studied in the second half of the 19th century. This paper presents the generalization of some geometric hyperbolic objects that are defined in the context of bicomplex analysis [3]. Building on this foundation, we have found many properties and applications of split complex numbers, for instance, we have introduced an algorithm to diagonalize a split complex real matrix [4]. As discussed earlier in this section, significant progress has been made in the split complex numbers in recent years. After the gradual development and maturation of the theory, with its unique advantages increasingly evident across a variety of domains, the researchers have gotten many breakthroughs in practical applications.

As one of the most important theorems in mathematical analysis, the Cauchy convergence criterion is essential in the process of studying the limit of series and function limit. And Cauchy convergence criterion encompasses various domains such as functions, sequences, integrals, and series. In this paper, the researcher has conducted an in-depth study of its proof and focuses on sufficient proof [5]. In this paper, the researchers develop some basic properties of these notions and find conditions in the space that the researchers benefited from which two summability methods coincide. Finally, we define Cauchy sequences in NNS and obtain the Cauchy-convergence criteria in these spaces [6]. In summary, Cauchy convergence criterion holds significant applications in various fields, including functions, sequences, calculus, and series. Moreover, it holds immense potential for further explorations and advancements in many realms.

Implement these research outcomes, split complex numbers allow one to express a meta-signal as the product of a code: this property solves the code ambiguity problem, thus leading to algorithms that are capable of effectively processing GNSS meta-signals [7]. Using the foundational concepts of split-complex number theory that have been established, the researchers can suggest the split-complex version of RSA cryptoalgorithm [8]. The researchers and their checks show that the errors occurring in calculation of parameter values and constants are very small when applying the Cauchy criterion for obtaining theoretical models of atmosphere Striking overvoltages [9]. In this article, a sparse signal recovery algorithm using Bayesian linear regression with convergence criterion is proposed. This improvement has brought about more reliable experimental findings [10]. After analyzing and summarizing the existing achievements, it is evident that there is a lack of relevant conclusions regarding the Cauchy convergence criterion of split complex plane. Therefore, this paper investigates the Cauchy convergence criterion of split complex plane and presents conclusions based on the proof of the Cauchy convergence criterion in the realm of real numbers.

## 2. The basic fundamental of split complex numbers $\mathbb{R}^{1,1}$

### 2.1 The introduction of split complex numbers $\mathbb{R}^{1,1}$

In this paper, the definition of split complex numbers can be written as

$$\mathbb{R}^{1,1} = \{\zeta = x + yj | x, y \in \mathbb{R}, j^2 = 1\}, \tag{1}$$

even though $\mathbb{R}^{1,1}$ and $\mathbb{C}$ have many similarities, there is also have different algebra structure. If $a, b \in \mathbb{C}$ and $a \cdot b = 0$, clearly, $a = 0$ or $b = 0$. Nevertheless, $\mathbb{R}^{1,1}$has zero divisors:

$$(1 + j)(1 - j) = 1 - j^2 = 0. \tag{2}$$

Thus it can defined that

$$j_+ = \frac{(1+j)}{2} \text{ or} j_- = \frac{(1-j)}{2}, \tag{3}$$

and the corollaries about $j_-$ and $j_+$ are as follows:

$$j_+ - j_- = j, j_+^2 = j_+, j_-^2 = j_-, j_+ + j_- = 1. \tag{4}$$

Let $\{j_+, j_-\}$ replace $\{1, j\}$ as base coordinate, therefore, some useful conclusions for $\mathbb{R}^{1,1}$ can be got :

$$\zeta = uj_+ + vj_-. \tag{5}$$

Where $u = x + y$ and$v = x - y$, and the multiplication can be simplified as:

$$\begin{aligned}(u_1 j_+ + v_1 j_-)(u_2 j_+ + v_2 j_-) &= u_1 u_2 (j_+)^2 + u_1 v_2 j_+ j_- + v_1 u_2 j_+ j_- + v_1 v_2 (j_-)^2 \\ &= u_1 u_2 j_+ + v_1 v_2 j_- \end{aligned} \tag{6}$$

Thus, as an algebra $\mathbb{R}^{1,1}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$. Precisely, the elements can be written as

$$\zeta = \alpha j_+ \text{ or } \zeta = \alpha j_-, \tag{7}$$

where $\alpha \in \mathbb{R}$. The definition and the basics have already been introduced, to get more conclusions for $\mathbb{R}^{1,1}$, more rules should be developed.

## 2.2 Partial order in split complex numbers $\mathbb{R}^{1,1}$

Let

$$\mathbb{R}_{+}^{1,1} = \{\zeta = uj_{+} + vj_{-} \mid u \geq 0, v \geq 0\} \text{ and } \mathbb{R}_{-}^{1,1} = \{\zeta = uj_{+} + vj_{-} \mid u \leq 0, v \leq 0\}, \quad (8)$$

$\alpha, \beta \in \mathbb{R}^{1,1}$, defined the partial order as follows:

$$\alpha \succcurlyeq \beta \Leftrightarrow \alpha - \beta \in \mathbb{R}_{+}^{1,1} \quad \alpha \preccurlyeq \beta \Leftrightarrow \alpha - \beta \in \mathbb{R}_{-}^{1,1}. \quad (9)$$

Positive split complex numbers and negative split complex numbers can be shown in Figure 1.

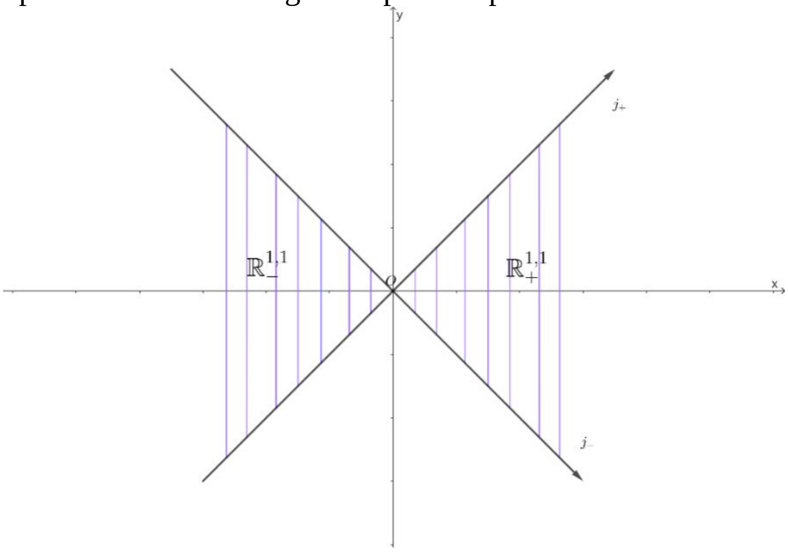


Figure 1: The figure about partial order

The geometric interpretation of Figure 1 is as follows: The positive split complex numbers are situated in the quarter plane denoted by $\mathbb{R}_{+}^{1,1}$, the negative split complex numbers are situated in the quarter plane denoted by $\mathbb{R}_{-}^{1,1}$. The other points cannot be called either positive or negative.

Based on the partial order, it can be defined interval

$$[\zeta, \omega]_{\mathbb{R}^{1,1}} = \{u \in \mathbb{R}^{1,1} \mid \zeta \preccurlyeq u \preccurlyeq \omega\}, \quad (10)$$

it is shown in Figure 2.

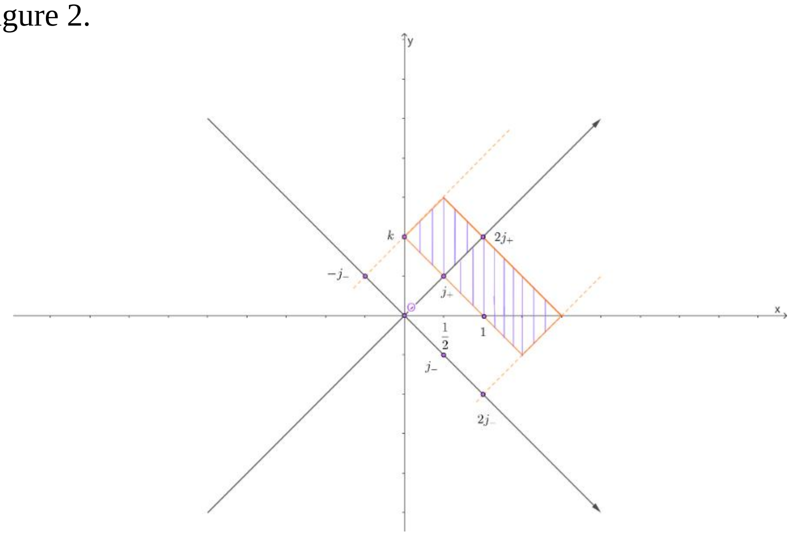


Figure 2: The figure about $[k, 2]_{\mathbb{R}^{1,1}}$

In Figure 2, the geometric significance of interval and partial order in split complex numbers $\mathbb{R}^{\mathbf{1,1}}$can be more intuitively illustrated. It can also show the general figure of the interval in split complex number $\mathbb{R}^{\mathbf{1,1}}$**.**

### 2.3 The definition of the convergence split complex sequence and moduli in split complex numbers $\mathbb{R}^{1,1}$

Building upon previous work, it can be stipulated that the moduli of split complex numbers :

$$|\zeta|_{\mathbb{R}^{1,1}} = |uj_+ + vj_-|_{\mathbb{R}^{1,1}} = |u|j_+ + |v|j_-. \tag{11}$$

In the split complex numbers, there is a sequence $\{\boldsymbol{\zeta_n}\}_{\boldsymbol{n}\in\mathbb{R}^{1,1}}$ converges to the split complex number$\boldsymbol{\zeta_0}$, if for any strictly positive split complex number$\boldsymbol{\varepsilon}$, there exists$\boldsymbol{N} \in \mathbb{N}$of any$\boldsymbol{n} \geq \boldsymbol{N}$there holds:

$$|\zeta_n - \zeta_0|_{\mathbb{R}^{1,1}} \prec \varepsilon. \tag{12}$$

we can deduce that

$$\zeta_n = u_{1n} \cdot j_+ + v_{2n} \cdot j_- \tag{13}$$

$$\zeta_0 = u_1 \cdot j_+ + v_2 \cdot j_- \tag{14}$$

$$\varepsilon = \varepsilon_1 \cdot j_+ + \varepsilon_2 \cdot j_-. \tag{15}$$

Equivalently, it can be stated that

$$|u_{1n} - u_1| < \varepsilon_1 \quad \text{and} \quad |v_{2n} - v_2| < \varepsilon_2. \tag{16}$$

## 3. Results

This section endeavors to provide a comprehensive discussion of the Cauchy convergence theorem of the split complex plane. This theorem, being a cornerstone in the realm of mathematical analysis, plays a vital role in ascertaining the convergence behavior of such sequences. By thoroughly examining the theorem, we aim to deepen our understanding of the intricate patterns and properties exhibited by separated complex sequences.

**Theorem [Cauchy convergence criterion of split complex plane]** Let $\{\zeta_n\}$be split complex numbers sequences and, the necessary and sufficient condition of $\{\zeta_n\}_{n\in\mathbb{N}}$converges to $\zeta_0$ is: $\zeta_n = u_n \cdot j_+ + v_n \cdot j_-$

For any strictly positive split complex number$\varepsilon$, $\varepsilon = \varepsilon_1 \cdot j_+ + \varepsilon_2 \cdot j_-$, there exists an index $N \in \mathbb{N}$such that $|\zeta_{n_1} - \zeta_{n_2}|_{\mathbb{R}^{1,1}} \prec \varepsilon$ whenever$n_1, n_2 \geq N$.

Proof:

(I) the proof of the necessity condition

Let $\{\zeta_n\}_{n\in\mathbb{R}^{1,1}}$ converges to$\zeta_0$, according to the content of the Theorem, for any strictly positive split complex number$\varepsilon$, there exists$N \in \mathbb{N}$, let$n_1, n_2 \geq N$,

$$\zeta_{n_1} = u_{n_1} \cdot j_+ + v_{n_1} \cdot j_-, \zeta_{n_2} = u_{n_2} \cdot j_+ + v_{n_2} \cdot j_-, \tag{17}$$

and

$$|\zeta_{n_1} - \zeta_0|_{\mathbb{R}^{1,1}} \prec \frac{\varepsilon}{2}, |\zeta_{n_2} - \zeta_0|_{\mathbb{R}^{1,1}} \prec \frac{\varepsilon}{2}. \tag{18}$$

According to the definition of the convergence split complex sequence conclude that

$$|u_{n_1} - u_1| < \frac{\varepsilon_1}{2} \quad , |v_{n_1} - v_2| < \frac{\varepsilon_2}{2}$$

$$\left|u_{n_2}-u_1\right|<\frac{\varepsilon_1}{2} \quad ,\left|v_{n_2}-v_2\right|<\frac{\varepsilon_2}{2}, \tag{19}$$

and because

$$\left|u_{n_1}-u_{n_2}\right| \leq \left|u_{n_1}-u_1\right|+\left|u_{n_1}-u_1\right|<\varepsilon_1$$
$$\left|v_{n_1}-v_{n_2}\right| \leq \left|v_{n_1}-v_1\right|+\left|v_{n_1}-v_1\right|<\varepsilon_2. \tag{20}$$

So it can concluded that

$$\begin{aligned} &|\zeta_{n_1}-\zeta_{n_2}|_{\mathbb{R}^{1,1}}=|\zeta_{n_1}-\zeta_0+\zeta_0-\zeta_{n_2}|_{\mathbb{R}^{1,1}} \\ &\leq (\left|u_{n_1}-u_1\right|+\left|u_{n_1}-u_1\right|)\cdot j_+ +(\left|v_{n_1}-v_1\right|+\left|v_{n_1}-v_1\right|)\cdot j_- \\ &\leq \varepsilon_1\cdot j_+ +\varepsilon_2\cdot j_- =\varepsilon \end{aligned} \tag{21}$$

The necessity condition can be proven.

(2) The proof of the sufficient condition

According to the given conditions, for any strictly positive split complex number$\varepsilon$, there exists an index $\boldsymbol{N} \in \mathbb{N}$ such that $|\zeta_{n_1}-\zeta_{n_2}|_{\mathbb{R}^{1,1}} \prec \varepsilon$ whenever$n_1, n_2 \geq N$. According to the definition of the convergence split complex sequence conclude that

$$\left|u_{n_1}-u_{n_2}\right|<\varepsilon_1$$
$$\left|v_{n_1}-v_{n_2}\right|<\varepsilon_2. \tag{22}$$

In this system of inequalities (22), $u_{n_1}, u_{n_2}, v_{n_1}, v_{n_2}$ are all real numbers. Obviously, Cauchy convergence criterion is holds for all real numbers, so the real numbers sequences $\{u_n\},\{v_n\}$ converge. By Theorem $\boldsymbol{\zeta_n=u_n\cdot j_+ +v_n\cdot j_-}$, the split complex numbers sequences also converge.

The sufficient condition can be proven.

## 4. Conclusion

This paper investigates the Cauchy convergence criterion in the split complex plane and gets the necessary and sufficient conditions for the convergence of the split complex numbers sequences. The difficulties brought by the zero divisors of split complex numbers have been overcome in this article, and the Cauchy convergence criterion by the decomposition property of split complex numbers. The conclusions presented in this paper further enrich the research on the convergence of sequences in bicomplex analysis, and solidify the theoretical foundation for the theories of functions of two complex variables and integration theory. This segment of content provides a research model and injects momentum into studies conducted in four-dimensional space.

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