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The Squeeze Principle of the Sequence of Hyperbolic Numbers

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Abstract: The hyperbolic numbers have an analogous composition with the complex numbers, which are composed by two real numbers, generating an exchangeable ring. That is to mean, the hyperbolic numbers can be viewed as the generalization of real numbers. In this article, we have proved the squeeze principle of the sequence of hyperbolic numbers. This article fills the gap in the field, and provides a new angle to prove the convergence of the sequence of the hyperbolic numbers. The result of this article constructs the basis for the future research on the hyperbolic numbers and the hyperbolic number plane. Meanwhile this article offers a new way to apply the squeeze principle of the sequence of hyperbolic numbers to the engineering.

1. Introduction

In the range of real numbers, the squeeze principle of sequence has been fully proved, and has good compatibility with other mathematical theorems, such as solving the limit of series with the binomial theorem. Ou Shufang provides a proof suitable for high school students, ensuring the theorem's application is within standard educational curricula. The article further explores the application of the Binomial Theorem in constructing sequences for the Squeeze Theorem, highlighting its utility in advanced mathematical problem-solving [1]. Besides, the squeeze principle also has a wide range of applications, for example, in the calculation of integral limits. Researchers have discussed methods for establishing inequalities using the squeeze principle in calculus, focusing on limit problems [2].

A step study of hyperbolic numbers is helpful to offer insights for educators and students seeking to apply the squeeze principle effectively. Furthermore, hyperbolic numbers can be used to solve more general mathematical problems, such as the Dirichlet problem [3]. Ozturk Iskender and Ozdemir Mustafa has determined affine transformations on Lorentzian plane using the set of hyperbolic numbers and given some examples on hyperbolical fractals. Also, they found some properties of hyperbolical reflections and rotations [4]. Through affine transformations, the hyperbolic number plane can form the iterated function systems. Tellez-Sanchez Y G and Bory-Reyes J investigate it and generalize the cookie-cutter Cantor sets in the real line to the hyperbolic number plane [5].

In present engineering construction, the hyperbolic numbers are also widely used. The future research on the hyperbolic number plane may provide more practical methods for mechanics, architecture and so on. Based on the lncosh cost function, Zhao Haiquan, Chen Jinsong and Wang Zhuonan proposed distributed complex logarithmic hyperbolic cosine algorithm and distributed enhanced complex logarithmic hyperbolic cosine algorithm. The algorithm exhibits enhanced robustness and better frequency estimation accuracy, making it a promising approach for power system applications [6].

In physics and biology, hyperbolic numbers can be used as new research tools. Based on hyperbolic numbers and bi-symmetric matrices, Petoukhov Sergey V reveals that the structural commonalities of different genetic biological systems is related to the harmonic progression 1/n and the harmonic mean [7]. Kulyabov S D, Korolkova V A and Gevorkyan N M use hyperbolic functions to represent two-dimensional Minkowski Spaces and Lorentz transformations, pointing out that the addition of velocities can be reduced to the addition of hyperbolic angles in hyperbolic numbers [8].

2. Preparations

2.1 The Partial Order

Supposing R is a binary relation on the set A, if R satisfies the following three properties, R is said to be a partial order relation on the set A, expressed as $\circ$ or $\pm$.

(1) Reflexivity

For any $x \in \mathrm{A}, xRx$.

(2)Anti symmetry

For any $x, y \in \mathrm{A}$,if xRy and yRx ,we have $x = y$.

(3)Transitivity

For any $x, y, z \in \mathrm{A}$,if xRy and yRz ,we have xRz.

2.2 The Hyperbolic Numbers

To achieve our results, we must define a special set of number, hyperbolic numbers, as follows.

$$\mathrm{D} := \{\delta = x + \mathbf{k}y : x, y \in \mathrm{R}\} \tag{1}$$

In this definition, the element $\mathbf{k}$ with the name "the hyperbolic unit" satisfies $\mathbf{k}^2 = 1$ and $\mathbf{k} \neq \pm 1$. The set D is obviously a commutative ring, with some useful features.

For

$$\delta_1 = x_1 + \mathrm{k}y_1 \tag{2}$$

$$\delta_2 = x_2 + \mathrm{k}y_2 \tag{3}$$

we have

$$\delta_1 + \delta_2 = \left(x_1 + \mathbf{k}y_1\right) + \left(x_2 + \mathbf{k}y_2\right) = \left(x_1 + x_2\right) + \mathbf{k}\left(y_1 + y_2\right) \tag{4}$$

and

$$\delta_1\delta_2 = \left(x_1 + \mathbf{k}y_1\right)\left(x_2 + \mathbf{k}y_2\right) = \left(x_1x_2 + y_1y_2\right) + \mathbf{k}\left(x_1y_2 + x_2y_1\right) \tag{5}$$

Example:

$$e := \frac{1+\mathbf{k}}{2}, e^{+} := \frac{1-\mathbf{k}}{2} \tag{6}$$

$$e+e^{+} = \frac{1+\mathbf{k}}{2} + \frac{1-\mathbf{k}}{2} = \left(\frac{1+1}{2}\right) + \mathbf{k}\left(\frac{1-1}{2}\right) = 1 \tag{7}$$

And

$$ee^{+} = \left(\frac{1+\mathbf{k}}{2}\right)\left(\frac{1-\mathbf{k}}{2}\right) = \frac{1+\mathbf{k}-\mathbf{k}-\mathbf{k}^2}{4} = 0 \tag{8}$$

$$\left(e^{+}\right)^2 = \left(\frac{1-\mathbf{k}}{2}\right)\left(\frac{1-\mathbf{k}}{2}\right) = \frac{2-2\mathbf{k}}{4} = \frac{1-\mathbf{k}}{2} = e^{+} \tag{9}$$

$$e+e^{+} = 1 \tag{10}$$

$$e-e^{+} = \mathbf{k} \tag{11}$$

In this example, the set $\{e, e^{+}\}$ composes the idempotent base of D.The e and e^{+} have the name of idempotent zero divisors, forming the whole ring. If we have $\delta = x + \mathbf{k}y \in \mathrm{D}$,we can transform it with e and e^{+}, $\delta = (x+y)e + (x-y)e^{+} =: ae + be^{+}$.

In set $\mathbb{D}$, all of the zero divisors must be the real multiples of e and e^{+} ,which are on the line $y = x$ and $y = -x$. As a result, δ can be a zero divisor if and only if $\delta = ae$ or $\delta = be^{+}$, $a, b \in \mathrm{R}^{*} - \{0\}$ .The hyperbolic interval is shown in Figure 1.

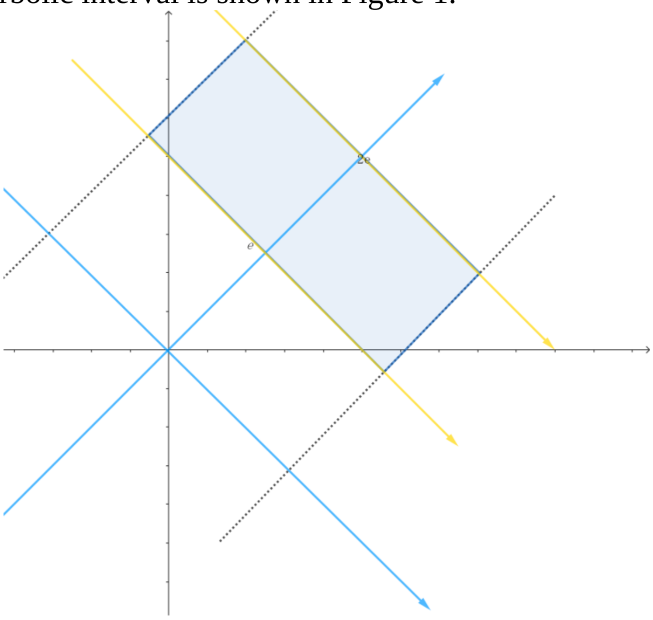

Figure 1: The hyperbolic interval

2.3 The Partial Order On $\mathbb{D}$

We can now have two similar sets, the set of non-negative hyperbolic numbers,

$$D^{+} := \{\delta = ae + be^{+} : x \geq 0, y \geq 0\} \tag{12}$$

and the set of non-positive hyperbolic numbers

$$D^{-} := \{\delta = ae + be^{+} : x \leq 0, y \leq 0\} \tag{13}$$

Geometrically, we can display the partial order on $\mathbb{D}$ like Figure 2.

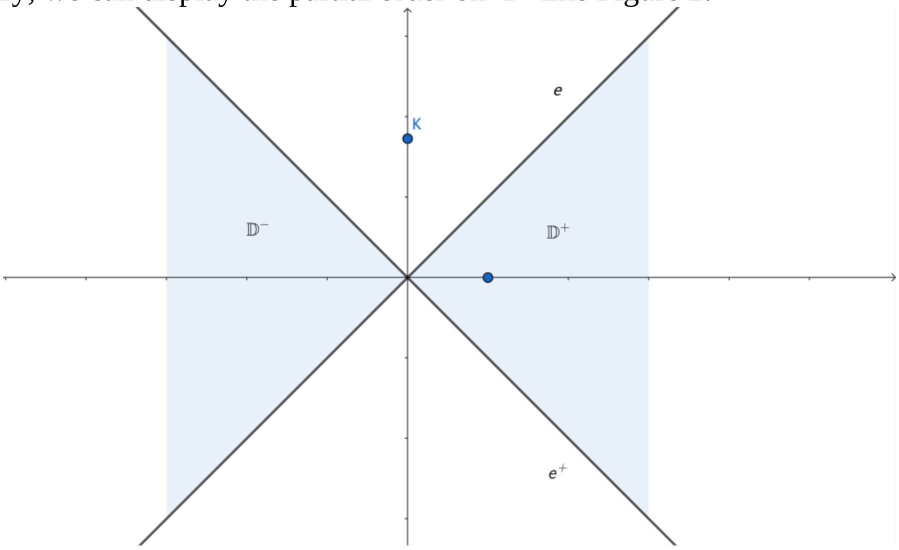

Figure 2: The description of positive and negative hyperbolic numbers

For $\delta = ae + be^{+}$ and $\delta_0 = a_0 e + b_0 e^{+}$, we have $\delta \circ \delta_0$ if and only if $a \leq a_0$ and $b \leq b_0$. Also, we have $\delta \pm \delta_0$ if and only if $a \geq a_0$ and $b \geq b_0$. Now we can describe the partial order on $\mathbb{D}$ in Figure 3.

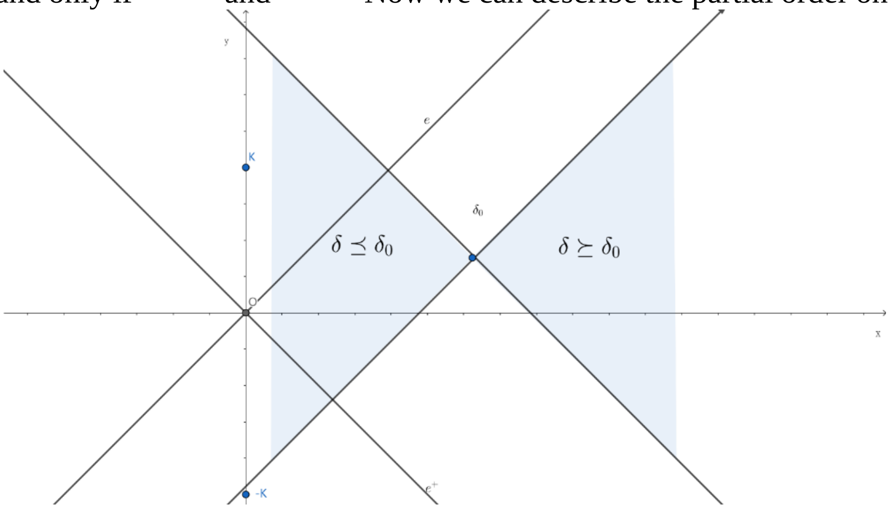

Figure 3: A partial order on $\mathbb{D}$

3. Results

As a result, we can prove the squeeze principle of the sequence of hyperbolic number.

Theorem3.1 (the squeeze principle of the sequence of hyperbolic number) For a sequence of hyperbolic number $\{\delta_n\}$,and for any positive integer n, if

$$\mu_n \preccurlyeq \delta_n \preccurlyeq \omega_n \tag{14}$$

then we can have the conclusion:

$$\text{If } \lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \omega_n = a, \text{ then } \lim_{n\to\infty} \delta_n = a. \tag{15}$$

The proof is as follows.

Proof:

For

$$\omega_n = \omega_{1n} e + \omega_{2n} e^+, \delta_n = \delta_{1n} e + \delta_{2n} e^+, \mu_n = \mu_{1n} e + \mu_{2n} e^+. \tag{16}$$

We have

$$\mu_n \preccurlyeq \delta_n \preccurlyeq \omega_n \tag{17}$$

so that we can just rewrite (15) into a new form

$$\mu_{1n} e + \mu_{2n} e^+ \preccurlyeq \delta_{1n} e + \delta_{2n} e^+ \preccurlyeq \omega_{1n} e + \omega_{2n} e^+. \tag{18}$$

Then according to the definition, we have

$$\begin{gathered} \mu_{1n} \preccurlyeq \delta_{1n} \preccurlyeq \omega_{1n}, \\ \mu_{2n} \preccurlyeq \delta_{2n} \preccurlyeq \omega_{2n}, \end{gathered} \tag{19}$$

$$\text{Besides} \lim_{n\to\infty} \omega_n = \lim_{n\to\infty} \left(\omega_{1n} e + \omega_{2n} e^+ \right) = a, \tag{20}$$

$$\Rightarrow \lim_{n\to\infty} \omega_{1n} = \lim_{n\to\infty} \omega_{2n} = a, \tag{21}$$

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \left(\mu_{1n} e + \mu_{2n} e^+ \right) = a, \tag{22}$$

$$\Rightarrow \lim_{n\to\infty} \mu_{1n} = \lim_{n\to\infty} \mu_{2n} = a, \tag{23}$$

Because of the squeeze theorem of the limit of real number sequence,

$$\lim_{n\to\infty} \delta_{1n} = \lim_{n\to\infty} \delta_{2n} = a. \tag{24}$$

We now draw the conclusion

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \delta_{1n} e + \delta_{2n} e^+ = a. \tag{25}$$

The geometric significance of the squeeze principle of the sequence of hyperbolic number is shown in Figure 4.

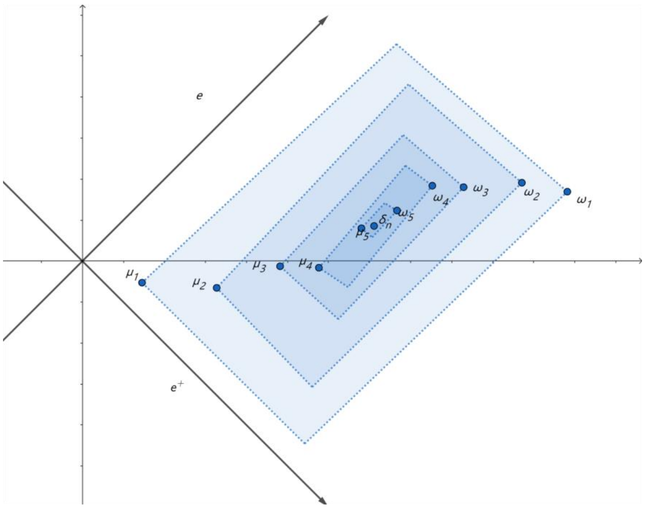

Figure 4: The geometric significance of the squeeze principle of the sequence of hyperbolic number

4. Conclusions

The hyperbolic numbers are composed by two real numbers, with a similar form with the complex numbers. But the hyperbolic numbers construct an exchangeable ring, containing idempotent zero factors. This article proves the squeeze principle of the sequence of hyperbolic numbers, further improves the theoretical basis of hyperbolic number analysis, and provides a new angle to prove the convergence of the sequence of hyperbolic numbers. It provides a direction for the future application of hyperbolic numbers in bridge construction, mechanical engineering and other fields.

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