Journal of Network Computing and Applications (2022)
Clausius Scientific Press, Canada
DOI: 10.23977/jnca.2022.070103
ISSN 2371-9214 Vol. 7 Num. 1

Research on the theory of matrix exchangeability

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Keywords: Matrix commutability, characteristic polynomial, minimal polynomial, linear transformation, fundamental matrix, diagonal matrix

Abstract: Based on the study of matrix theory, the conditions of matrix commutability are given and some properties of matrix commutability are obtained. The relationship between matrix characteristic polynomial and minimum polynomial, matrix and linear transformation relationship and other knowledge are used.

1. Matrix exchangeability

1.1 Definition of matrix exchangeability

If A and B are two n-order square matrices, if AB = BA, then a and B can be exchanged.

We mainly consider that we know a matrix A and find all the matrices that can be exchanged with it Therefore, we define the whole matrix that can be exchanged between all and a, which is recorded as C (A) Note that F (A) is all generated by the polynomial of matrix A.

Theorem 1: For any a, ab = Ba is equivalent to (A-AE) B = B (A-AE).

In fact, (A-AE) B = B (A-AE) can be transformed into AB-B = BA-B, that is AB = BA

2. The Internal Relationship between F(A) and C(A) in Matrix Exchangeability

Theorem 2.1: Sets A to an n-order square matrix of domain P.

[1]. Subspaces of spaces composed of all n-order square matrices on the domain P of all components of matrices that can be exchanged between all and A.

[2]. F(A) is a linear space generated by the polynomial of matrix A, and F(A) is included in C(A).

The proof process: [1] because en is in C (A), C (A) is non null Let B, C in C (A), then BA = AB, CA = AC, so A (B + C) = (B + C) A, ie B + C in C (A), (KB) A = A (KB), ie kb in C (A) Corroborated [2] Since the polynomials of a matrix are exchangeable with the matrix, the conclusion holds. Further elucidation of their two connections follows

Theorem 2.2: Let A be an n-class nonzero matrix over the number domain P, the minimum number of polynomials m (x) for A be r, then the set $V = \{f(A)|f(x) \in P[X]\}$ with respect to the additive sum of matrices constitutes the linear space in the R dimension, and $E,A,\cdots,A^{r-1}$ is a set of bases for V.

The proof process: $f(x) = x \in V$, So V is non null. Arbitrary $f(x)$, $g(x) \in P[X]$, $k \in P$, $(f(x)+g(x)) \in P[X]$, $kg(x) \in P[X]$, So V satisfies additive and number by closure So V is the subspace of M_n (P), that is, the additive and number multiplication of V about the matrix constitutes the linear space It is demonstrated below that V constitutes a linear space in the R dimension, and $E,A,\cdots,A^{r-1}$ is a set of bases of V.

a) Suppose that there exists $k_0,\cdots,k_{r-1}$, that is not fully zero such that $k_0E+k_1A+\cdots+k_{r-1}A^{r-1}=0$, Then the polynomial $k_0+k_1x+\cdots+k_{r-1}x^{r-1}$ is a zeroing polynomial that is strictly lower in number than the minimal polynomial, which contradicts the definition of the minimal polynomial, so there must be $k_0=\cdots=k_{r-1}=0$, $E,A,\cdots,A^{r-1}$ linear independence.
b) For an arbitrary $f(x) \in P[x]$, by the band residual division, there exists q(x), r(x) such that $f(x) = q(x)m(x) + r(x)$, Where r(x)=0 or $\partial(r(x))<\partial(m(x))$, let $r(x)=b_{r-1}x^{r-1}+\cdots+b_1x+b_0$, and thereby.
$f(A)=q(A)m(A)+r(A)=b_{r-1}A^{r-1}+\cdots+b_1A+b_0E$. This illustrates that an arbitrary matrix in V can all be linearly expressed by $E,A,\cdots,A^{r-1}$. So $E,A,\cdots,A^{r-1}$ is a set of bases of V, thus dimV = r.

3. Application of the characteristic polynomial equal to the smallest polynomial

Now give a more general conclusion:

Theorem 3.1: Let A,B be linear transformations of the n-dimensional linear space over the number domain P, and it is known that the characteristic polynomial of A equals the smallest polynomial, both $f(\lambda)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n$,

[1]. The matrix of A under a certain set of bases $\alpha_1,\cdots\alpha_{n-1}$ is:

$$\boldsymbol{A}=\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{a_n} \\ \mathbf{1} & \mathbf{0} & \ddots & \mathbf{0} & -\boldsymbol{a_{n-1}} \\ \mathbf{0} & \mathbf{1} & \ddots & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} & -\boldsymbol{a_2} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} & -\boldsymbol{a_1}\end{bmatrix}$$

[2]. Versus that in [1] α_1, there is α_1, $A\alpha_1,\cdots$, $\boldsymbol{A^{n-1}}\alpha_1$ also based.

[3]. If AB = BA, then there exists $l_0,\cdots,l_{n-1}\in P$, such that $\mathbf{B}=\sum_{j=0}^{n-1}\boldsymbol{l_j A^j}$, that is, F (A) = C (A) (that is, the linear transformations exchangeable with A are all A polynomials)

The proof procedure:

[1]. Let any $e_1,\cdots,e_n$ be a set of bases for V, let the matrix of the linear transformation A under this set of bases be B, then the characteristic polynomial of B is equal to the smallest polynomial equal to the last invariant factor, so the invariant factor of B is $1,\cdots,1$, $f(\lambda)$.In considering $\lambda E-A$ shows that the determinant factor of matrix A is $1,\cdots,1$, $f(\lambda)$, Then the invariant factor of a is also $1,\cdots,1$, $f(\lambda)$.So A, B have the same invariant factor, so a, B are similar, and there exists a reversible array P such that p-1ap = B, so there must be a set of bases $\alpha_1,\cdots\alpha_{n-1}$ such that A in the matrix under $\alpha_1,\cdots\alpha_{n-1}$ is A.

[2]. Because A has a matrix under some set of bases α_1, ··· α_{n-1}, there is A. so $\boldsymbol{A}\alpha_1=\alpha_2$,$\boldsymbol{A}\alpha_2=\alpha_3,\cdots,\boldsymbol{A}\alpha_{n-1}=\alpha_n$, that is, $\alpha_1,\boldsymbol{A}\alpha_1,\boldsymbol{A}^{n-1}\alpha_1$ also based.

[3]. Since AB = BA, for any integer k, $\boldsymbol{A}^k B = B\boldsymbol{A}^k$. For an arbitrary $\alpha \in V$, let $\alpha=\sum_{i=0}^{n-1} k_i\, \boldsymbol{A}^i \alpha_1$, $B\alpha_1=\sum_{j=0}^{n-1} l_j\, \boldsymbol{A}^j \alpha_1$,so:
$B\alpha = B\sum_{i=0}^{n-1} k_i\, \boldsymbol{A}^i \alpha_1 = \sum_{i=0}^{n-1} k_i\, B\boldsymbol{A}^i \alpha_1 = \sum_{i=0}^{n-1} k_i\, \boldsymbol{A}^i \mathcal{B}\alpha_1 = \sum_{i=0}^{n-1} k_i\, \boldsymbol{A}^i (\sum_{j=0}^{n-1} l_j\, \boldsymbol{A}^j \alpha_1) = \sum_{j=0}^{n-1} l_j\, \boldsymbol{A}^j (\sum_{i=0}^{n-1} k_i\, \boldsymbol{A}^i \alpha_1) = \sum_{j=0}^{n-1} l_j\, \boldsymbol{A}^j \alpha$.

Because the arbitrary nature of α is known, $B = \sum_{j=0}^{n-1} l_j\, \boldsymbol{A}^j$ is F (A) =C(A).

4. Applications of exchangeable matrices for dealing with certain problems

Theorem 4.1: Let V be an n-dimensional linear space on the complex domain, A, B be linear transformation on V, satisfying AB = BA. then A, B have A common eigenvector.

The proof procedure: As known from the knowledge of the invariant subspace, both the feature subspaces of A are invariant subspaces of B, now take V_λ Is the eigenvalue of A, λ of the feature subspace, then V_λ is the invariant subspace of B, $B|V_\lambda$ is the V_λ A linear transformation, because it is considered in the complex domain, the linear transformation certainly has eigenvalues, let μ corresponds to the characterized subspace $V_\mu \subset V_\lambda$. Any nonzero vector $\alpha \in V_\mu$, that $\alpha \in V_\lambda$, so there is $B\alpha=\mu\alpha$, there is also $A\alpha=\mu\alpha$, α is the common eigenvector of A, B

[1] has the above theorem, and the following can also be obtained with ease: let v be the n-dimensional linear space on the complex domain, A, B be the linear transformation on V satisfying AB = BA, and if a has s mutually distinct eigenvalues, then A, B have at least s common and linearly independent eigenvectors demonstrating that the methods are generally similar, here is actually generalizing the number of eigenvalues.

There are other forms of the above conclusion, such as: let v be the n (which is odd) dimensional linear space on real domain, A, B be the linear transformation on V satisfying AB = BA, then A, B have common eigenvectors This general idea is similar at the time of demonstration, but requires to come to a conclusion that the dimension of the root subspace $ker(\mathcal{A} - \lambda_i \mathcal{E})^{r_i}$ equals an algebraic weight of λ_i, r_i (which holds true for any i), such that taking V_λ,V_μ above guarantees a eigenvalue and, in turn, allows the proof to continue

Theorem 4.2: Let A, B be two arrays of order n over the complex domain, AB = BA, then A, B have common eigenvectors.

The proof procedure: by theorem 4.1, the above conclusion is the matrix language and obviously holds note it is more difficult to prove this theorem directly because the matrix is free of invariant subspace.

Theorem 4.3: Let A, B be a matrix of order n on the complex domain, AB = BA, then there exists a reversible matrix P-1BP such that P-1AP is simultaneously an upper triangular matrix with P.

The proof procedure: For n as mathematical induction. With n = 1, the conclusion clearly holds suppose that the conclusions hold for n-1, considering the case of order n.

Because A, B are matrices of order n on the complex domains, AB = BA, by theorem 4.2, A, B have common eigenvectors, denoted by α_1, let $A\alpha_1=\lambda\alpha_1$,$B\alpha_1=\mu\alpha_1$, put α_1 is extended by $\alpha_1,\cdots,\alpha_n$, set of bases C^n, denoted $P_1=(\alpha_1,\cdots,\alpha_n)$ of C^n, which is a reversible matrix that satisfies

$$P_1^{-1}AP_1 = \begin{bmatrix} \lambda & \alpha' \\ 0 & A_{n-1} \end{bmatrix},\ P_1^{-1}BP_1 = \begin{bmatrix} \mu & \beta' \\ 0 & B_{n-1} \end{bmatrix},$$

AB=BA, so$P^{-1}APP^{-1}BP=P^{-1}BPP^{-1}AP$,

$$\begin{bmatrix} \lambda & \alpha' \\ 0 & A_{n-1} \end{bmatrix}\begin{bmatrix} \mu & \beta' \\ 0 & B_{n-1} \end{bmatrix} = \begin{bmatrix} \mu & \beta' \\ 0 & B_{n-1} \end{bmatrix}\begin{bmatrix} \lambda & \alpha' \\ 0 & A_{n-1} \end{bmatrix},$$

So:

$$\begin{bmatrix} \lambda\mu & \lambda\beta' + \alpha' B_{n-1} \\ 0 & A_{n-1}B_{n-1} \end{bmatrix} = \begin{bmatrix} \lambda\mu & \mu\alpha' + \beta' B_{n-1} \\ 0 & B_{n-1}A_{n-1} \end{bmatrix},$$

$A_{n-1}B_{n-1} = B_{n-1}A_{n-1}$, Using the inductive assumption, there is a reversible array Q of n-1 levels such that $Q^{-1}A_{n-1}Q$ is simultaneously an upper triangular matrix with $Q^{-1}B_{n-1}Q$, taken

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix},$$

So $Q_1^{-1}P_1^{-1}AP_1Q_1 = \begin{bmatrix} \lambda & \alpha' Q \\ 0 & Q^{-1}A_{n-1}Q \end{bmatrix}$ and $Q_1^{-1}P_1^{-1}BP_1Q_1 = \begin{bmatrix} \mu & \beta' Q \\ 0 & Q^{-1}B_{n-1}Q \end{bmatrix}$ all were upper triangular arrays. So $P = P_1Q_1$.

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