

The Hilbert Transforms of Complex Shannon Wavelet and Marr Wavelet

YANG Xiuzhu

College of Science, China Three Gorges University, Yichang 443002, China

XiuzhuYang@hotmail.com.

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Abstract: The Hilbert transform of complex Shannon wavelet is firstly calculated by using the Residue Theorem. The Hilbert transform of Marr wavelet is then computed by applying the Hilbert transform of Gaussian functions. Since these two kinds of wavelets are relatively important in practice, the calculation of the Hilbert transforms of these two kinds of wavelets could be useful for designing of the wavelets which form a pair of Hilbert transform.

1. Introduction

Wavelets are very useful mathematical tool, which is widely used in many fields such as signal analysis, data compression, operator analysis, partial differential equation solving, solid mechanics^[1]. The two common wavelets involved in this article also have their own important applications. Shannon wavelets are analytically defined and infinitely differentiable, with a very short boundary in the frequency domain. Therefore, the frequency can be well decomposed in a narrow band, and the coefficients of any derivative of it can be defined analytically, while for other wavelet families, only the low-order derivatives can be numerically calculated^[1]. The special properties of Shannon wavelets make it very useful in solving calculus equations^[2]. The general form of Marr wavelets is defined by the n-th-order derivative of Gauss function, so it inherits the properties of Gauss wavelet that has both time domain and frequency domain attenuation. Marr wavelets have applications in hydrology^[3].

Wavelets have good properties and wide applications, but it also has some shortcomings of its own, such as sensitivity to translation, weak directionality in digital image processing, and lack of phase information. Researchers found that biorthogonal wavelets can solve some of the above problems, so some people have launched related research. Ozkaramanl et al. first studied the phase conditions that the scale functions corresponding to the two wavelets forming the Hilbert transform pair need to meet, and then based on this they constructed the Hilbert transform pair of the biorthogonal wavelets base^[4-5]. The Hilbert transform of wavelets can help wavelet overcome its own shortcomings. However, in the research, the Hilbert transform of Harr wavelets is mostly used as a typical research and application, and there are few researches on the Hilbert transform of other wavelets. Harr wavelets are too simple and discontinuous, and do not have many good properties, which limits the application of its Hilbert transform. So this paper studies the Hilbert transform of two common wavelets, gives their Hilbert transform expressions, and finds that they have simple

forms and good properties. This provides more choices for researchers who need to use the Hilbert transform of wavelets in subsequent research.

Section 2 gives the definition of two common wavelets, the definition of Hilbert transform and some necessary notations. Section 3.1 introduces the process of obtaining the Hilbert transform of the complex Shannon wavelets, and then gives the specific form of the Hilbert transform of the complex Shannon wavelets. Section 3.2 introduces the process of obtaining the Hilbert transform of Marr wavelets, and gives the specific form of the Hilbert transform of Marr wavelets. Finally, the work of this paper is summarized in Section 4.

2. Preliminaries

The mother wavelet expression of complex Shannon wavelet is

$$\psi(t) = \frac{e^{4\pi it} - e^{2\pi it}}{2\pi it} \quad [1]. \quad (1)$$

Marr wavelets are also called Mexican Hat wavelets, and its mother wavelet expression is

$$\psi(t) = \frac{2}{\sqrt{3}} \pi^{-3/4} (1-t^2) e^{-t^2/2} \quad [6]. \quad (2)$$

The Gauss function in this paper follows the form in [6],

$$G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \quad (3)$$

In this paper, we discuss the Hilbert transformation on the square integrable function space $L^2(R)$, that is, the two norm of f is $\|f\|_2 = \left(\int_R |f(x)|^2 dx \right)^{1/2}$, and $L^2(R) = \{f \mid \|f\|_2 < \infty\}$. For any $f \in L^2(R)$, the Hilbert of f is defined at $x \in R$ as

$$Hf(x) = \frac{1}{\pi} \cdot p.v. \int_R \frac{f(t)}{x-t} dt = \frac{1}{\pi} \lim_{\substack{\varepsilon \rightarrow 0^+ \\ N \rightarrow \infty}} \int_{\varepsilon \leq |x-t| \leq N} \frac{f(t)}{x-t} dt, \quad (4) \text{ where } p.v. \text{ represents the}$$

principal value integral.

3. Main Results

3.1 Hilbert Transform of Complex Shannon Wavelets

From the definition of Hilbert transform combined with formula (1), it can be seen that the Hilbert transform of the complex Shannon wavelets is

$$H\psi(x) = \frac{1}{\pi} \cdot p.v. \int_R \frac{e^{4\pi it} - e^{2\pi it}}{2\pi it \cdot (x-t)} dt. \quad (5)$$

Slightly deforming formula (5), we can get

$$H\psi(x) = \frac{i}{2\pi^2} \cdot p.v. \int_R \frac{e^{4\pi it} - e^{2\pi it}}{t \cdot (t-x)} dt = \frac{i}{2\pi^2} \cdot \left(p.v. \int_R \frac{e^{4\pi it}}{t \cdot (t-x)} dt - p.v. \int_R \frac{e^{2\pi it}}{t \cdot (t-x)} dt \right). \quad (6)$$

Let's look at the two integral parts first. Let $B = p.v. \int_R \frac{e^{4\pi it}}{t \cdot (t-x)} dt$, $C = p.v. \int_R \frac{e^{2\pi it}}{t \cdot (t-x)} dt$. Setting auxiliary function $p(z) = \frac{1}{z(z-x)}$ for B , in addition $\lim_{z \rightarrow \infty} |p(z)| = 0$, these show that B

satisfies the conditions of residue theorem. According to the residue theorem, it is easy to get $B = \pi i \{ \text{Re } sf(0) + \text{Re } sf(x) \} = \pi i \left\{ \frac{1}{x} - \frac{e^{4\pi i x}}{x} \right\}$. In the same way, it's not hard to get

$C = \pi i \left\{ \frac{1}{x} - \frac{e^{2\pi i x}}{x} \right\}$. Substituting B and C back into formula (6) can obtain

$H\psi(x) = \frac{i}{2\pi^2} \cdot \left(\pi i \left(\frac{1}{x} - \frac{e^{4\pi i x}}{x} \right) - \pi i \left(\frac{1}{x} - \frac{e^{2\pi i x}}{x} \right) \right)$. Then the Hilbert transform of the mother wavelet

of complex Shannon wavelets is easily obtained by merging similar terms and other operations to

$$H\psi(x) = -\frac{e^{4\pi i x} - e^{2\pi i x}}{2\pi x}. \quad (7)$$

3.2 Hilbert Transform of Marr Wavelets

The mother wavelet of Marr wavelets is actually the second derivative of Gauss function. The Hilbert transform of the Gauss function is known as

$$H \left\{ e^{-\alpha t^2} \right\} (x) = \frac{1}{\sqrt{\alpha\pi}} e^{-\alpha x^2} \int_0^{2\alpha x} e^{y^2/4\alpha} dy \quad [7]. \quad (8)$$

By slightly deforming the mother wavelet of Marr wavelets, we can get $\psi(t) = \frac{2}{\sqrt{3}} \pi^{-1/4} (1-t^2) e^{-t^2/2} = \frac{2}{\sqrt{3}} \pi^{-1/4} e^{-t^2/2} - \frac{2}{\sqrt{3}} \pi^{-1/4} t^2 e^{-t^2/2}$. Let $f(t) = \frac{2}{\sqrt{3}} \pi^{-1/4} e^{-t^2/2}$ and $g(t) = \frac{2}{\sqrt{3}} \pi^{-1/4} t^2 e^{-t^2/2}$. So the Hilbert transform of mother wavelet of Marr wavelet is

$$H\psi(x) = Hf(x) - Hg(x). \quad (9)$$

Look at the part of $Hf(x)$ first, substituting formula (8) into $Hf(x)$, there is

$$Hf(x) = 2\sqrt{\frac{2}{3}} \pi^{-3/4} e^{-x^2/2} \int_0^x e^{y^2/2} dy. \quad (10)$$

In order to get $Hg(x)$, consider $g(t)$ as a function obtained by taking the Gaussian function coefficient as a parameter to obtain the first derivative. $g(t) = \frac{2}{\sqrt{3}} \pi^{-1/4} t^2 e^{-t^2/2} = -\frac{2}{\sqrt{3}} \pi^{-1/4} \frac{\partial}{\partial \lambda} e^{-\lambda t^2} \Big|_{\lambda=1/2}$. So

we have $Hg(x) = -\frac{2}{\sqrt{3}} \pi^{-1/4} \frac{\partial}{\partial \lambda} H \left\{ e^{-\lambda t^2} \right\} \Big|_{\lambda=1/2} (x)$. Substituting formula (8) into the above formula, and

then according to the chain rule of derivation, we can get

$$Hg(x) = -\frac{2}{\sqrt{3}} \pi^{-1/4} \left[-\frac{1}{2\lambda} \frac{1}{\sqrt{\lambda\pi}} e^{-\lambda x^2} \int_0^{2\lambda x} e^{y^2/4\lambda} dy - \frac{x^2}{\sqrt{\lambda\pi}} e^{-\lambda x^2} \int_0^{2\lambda x} e^{y^2/4\lambda} dy + \frac{1}{\sqrt{\lambda\pi}} e^{-\lambda x^2} \frac{\partial}{\partial \lambda} \int_0^{2\lambda x} e^{y^2/4\lambda} dy \right] \Big|_{\lambda=1/2}.$$

According to the properties of integrals containing parameters,

$\frac{\partial}{\partial \lambda} \int_0^{2\lambda x} e^{y^2/4\lambda} dy = 2xe^{\lambda x^2} - \int_0^{2\lambda x} \frac{y^2 e^{y^2/4\lambda}}{4\lambda^2} dy$. So obviously

$$Hg(x) = 2\sqrt{\frac{2}{3}} \pi^{-3/4} e^{-x^2/2} \left[(x^2 + 1) \int_0^x e^{y^2/2} dy + \int_0^x y^2 e^{y^2/2} dy \right] - 4\sqrt{\frac{2}{3}} \pi^{-3/4} x. \quad (11)$$

Substituting formula (10) and formula (11) back to formula (9) to obtain the Hilbert transform expression of Marr wavelets as

$$H\psi(x) = 2\sqrt{\frac{2}{3}}\pi^{-3/4} \left[2x - \int_0^x (x^2 + y^2) e^{(y^2-x^2)/2} dy \right]. \quad (12)$$

4. Conclusion

Two Hilbert transforms of wavelets with good performance and practical application significance are introduced in this article. They are the complex Shannon wavelets and Marr wavelets, and the specific forms of their Hilbert transform are given. This provides a reference for researchers who follow-up on the Hilbert transform of wavelets, and has certain scientific significance.

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