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Iterative properties on flow

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Abstract: In this paper, we consider a continuous flow $\varphi: \mathcal{R} \times X \to X$,where X is a compact metric space,and we prove that for any positive integer N, φ is distributional chaotic if and only if φ^N is distributional chaotic;φ is Li-Yorke chaotic if and only if φ^N is Li-Yorke chaotic.

1. Introduction

In 1975, Li and Yorke first gave the definition of chaos (see [1]) , the definition opened the door on researching chaos, many scholars began to explore the chaos and give the different notions and concepts of chaos .In 1994, Schweizer and Smital defined a new chaos named distributional chaos (see[2,3]) . The scholar's effort is to clarify the essence of the complexity of dynamical systems. Nowadays to investigate the chaotic behavior of dynamical systems has become a hot subject.

In this paper, we main obtain the following results: Let (X, d) be a compact metric space with metric d, write $\mathbb{R} = (-\infty, +\infty)$. Let $\varphi: \mathcal{R} \times X \to X$ be a continuous flow.

(1) For any integer $N > 0$, φ is distributional chaotic if and only if φ^N is distributional chaotic.
(2) For any integer $N > 0$, φ is Li-Yorke chaotic if and only if φ^N is Li-Yorke chaotic.

2. Preliminarier

Let (X, d) be a compact metric space with metric d, write $\mathbb{R} = (-\infty, +\infty)$. We call $\varphi: \mathcal{R} \times X \to X$ is a continuous flow if φ satisfies the following conditions£»

(1) $\varphi(0, x) = x, \forall x \in X$.
(2) $\varphi(t, \cdot): X \to X, \forall t \in \mathbb{R}$ is homeomorphism.
(3) $\varphi(t, \varphi(s, x)) = \varphi(s + t, x), \forall s, t \in \mathbb{R}$.

Given $k \in \mathbb{R}$, we define $\varphi^k: \mathbb{R} \times X \to X$,where $\varphi^k(t, x) = \varphi(kt, x), \forall x \in X$ (refer to [4] for more details) .

The product metric ρ on the product space $X \times X$ is defined by

$$\rho((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\}$$

for any $(x, y), (x', y') \in X \times X$.

Definitions 2.1 φ is said to be Li-Yorke chaotic if there exists an uncountable set $D \subset X$ such that for any pair $(x, y) \in D \times D$ with $x \neq y$,

(1) $\liminf_{t\to\infty} d\left(\varphi(t,x), \varphi(t,y)\right) = 0$; (2) $\limsup_{t\to\infty} d\left(\varphi(t,x), \varphi(t,y)\right) > 0$.

Sometimes (x, y) is said to be a Li-Yorke pair of φ.

Definitions 2.2 For any real number $s > 0, x, y \in X$, let

(1) $\underline{F}_{xy}(s) = \liminf_{t\to\infty} \frac{1}{t}\int_0^t \chi_{[0,s]}\left(d(\varphi(t,x), \varphi(t,y))\right)dt$.

(2) $\overline{F}_{xy}(s) = \limsup_{t\to\infty} \frac{1}{t}\int_0^t \chi_{[0,s]}\left(d(\varphi(t,x), \varphi(t,y))\right)dt$.

Where $\chi_A(x)$ is 1 if $x \in A$, and $\chi_A(x)$ is 0 if $x \notin A$. Obviously $\underline{F}_{xy}(s)$ and $\overline{F}_{xy}(s)$ are both nondecreasing functions. We call $(x, y) \in X \times X$ is a pair displaying distributional chaos if

(1) $\underline{F}_{xy}(\alpha) = 0$, for some $\alpha > 0$; (2) $\overline{F}_{xy}(s) = 1$, for any $s > 0$.

φ is said to display distributional chaotic if there exists an uncountable set $D \subset X$ such that any two different points in D is a pair displaying distributional chaos.

From the above definitions, we can see that any map displaying distributional chaos must be Li-Yorke chaotic.

For simplicity, let

$$\varepsilon_t(\varphi, x, y, s) = \int_0^t \chi_{[0,s]}\left(d(\varphi(t,x), \varphi(t,y))\right)dt.$$

$$\underline{F}(\varphi, x, y, s) = \liminf_{t\to\infty} \frac{1}{t}\varepsilon_t(\varphi, x, y, s).$$

$$\overline{F}(\varphi, x, y, s) = \limsup_{t\to\infty} \frac{1}{t}\varepsilon_t(\varphi, x, y, s).$$

3. Lemmas

In order to prove the main theorems , at first we show some lemmas.

Lemma 3.1 Let $\varphi: \mathcal{R} \times X \to X$ be a continuous flow,$x, y \in X$. For any positive integer $N > 0$, and $s > 0$, we have

(1) If $\underline{F}(\varphi, x, y, s) = 0$, then$\underline{F}(\varphi^N, x, y, s) = 0$.

(2) If $\overline{F}(\varphi, x, y, s) = 1$, then$\overline{F}(\varphi^N, x, y, s) = 1$.

Proof (1) If $\underline{F}(\varphi, x, y, s) = 0$,then there is an increasing sequence $\{t_i\}$ such that when $i \to \infty$,

$$\lim_{i\to\infty} \frac{1}{t_i}\varepsilon_{t_i}(\varphi, x, y, s) = 0.$$

Put

$$m_i = \frac{t_i}{N}.$$

then for each i,

$$\varepsilon_{m_i}(\varphi^N, x, y, s) = \varepsilon_{t_i}(\varphi, x, y, s).$$

It follows that for $i \to \infty$,

$$\lim_{i\to\infty} \frac{1}{t_i}\varepsilon_{m_i}(\varphi^N, x, y, s) = 0.$$

and further

$$\lim_{i\to\infty} \frac{N}{t_i}\varepsilon_{m_i}(\varphi^N, x, y, s) = 0.$$

This gives for $i \to \infty$,

$$\lim_{i\to\infty}\frac{1}{m_i}\varepsilon_{m_i}(\varphi^N, x, y, s) = 0..$$

Therefore

$$\underline{F}\left(\varphi^N, x, y, s\right) = 0.$$

(2) If $\overline{F}(\varphi, x, y, s) = 1$, then there is an increasing sequence $\{t_i\}$ such that when $i \to \infty$,

$$\lim_{i\to\infty}\frac{1}{t_i}\varepsilon_{t_i}\left(\varphi, x, y, s\right) = 1.$$

Let

$$\delta_{t_i}\left(\varphi, x, y, s\right) = \ell\left\{t : d\left(\varphi(t, x), \varphi(t, y)\right) \geq s, 0 \leq t < t_i\right\}.$$

where $\ell\{t : d\left(\varphi(t, x), \varphi(t, y)\right) \geq s, 0 \leq t < t_i\}$ denotes the Lebesque measure

$$\{t : d\left(\varphi(t, x), \varphi(t, y)\right) \geq s, 0 \leq t < t_i\}.$$

Because for each t_i,

$$\frac{1}{t_i}\delta_{t_i}\left(\varphi, x, y, s\right) + \frac{1}{t_i}\varepsilon_{t_i}\left(\varphi, x, y, s\right) = 1.$$

We have

$$\lim_{i\to\infty}\frac{1}{t_i}\delta_{t_i}\left(\varphi, x, y, s\right) = 0.$$

Put $m_i = \frac{t_i}{N}$. By an argument similar to that given above , we get that

$$\lim_{i\to\infty}\frac{1}{m_i}\delta_{m_i}\left(\varphi^N, x, y, s\right) = 0..$$

and further

$$\lim_{i\to\infty}\frac{1}{m_i}\varepsilon_{m_i}\left(\varphi^N, x, y, s\right) = 1 - \lim_{i\to\infty}\frac{1}{m_i}\delta_{m_i}\left(\varphi^N, x, y, s\right) = 1.$$

This means that

$$\overline{F}\left(\varphi^N, x, y, s\right) = 1.$$

Lemma 3.2 Let $\varphi : \mathcal{R} \times X \to X$ be a continuous flow, $x, y \in X$, $N > 0$, then the following results hold:

(1) If for $s > 0$, $\underline{F}(\varphi^N, x, y, s) = 0$, then there exists $p > 0$ such that $\underline{F}(\varphi, x, y, p) = 0$.

(2) If $\overline{F}(\varphi^N, x, y, s) = 1$ for all $s > 0$,then $\overline{F}(\varphi, x, y, p) = 1$ or all $p > 0$.

Proof (1) If for $s > 0$, $\underline{F}(\varphi^N, x, y, s) = 0$,then there exists an increasing sequence $\{t_i\}$ such that when $i \to \infty$,

$$\lim_{i\to\infty}\frac{1}{t_i}\varepsilon_{t_i}(\varphi^N, x, y, s) = 0.$$

Since X is compact, and the map $\varphi:[0,N]\times X\to X$ is uniform continuous, hence for fixed $s>0$, there exists $p>0$ such that for $\mu,\nu\in X$ and each $t\in[0,N], d(\varphi(t,\mu),\varphi(t,\nu))\geq p$ implies $d(\varphi(N,\mu),\varphi(N,\nu))\geq s$. So we have

$$N\delta_{t_i}(\varphi^N,x,y,s)\leq\delta_{Nt_i}(\varphi,x,y,p).$$

Put $m_i=Nt_i$. Then we have

$$\frac{1}{m_i}\varepsilon_{m_i}(\varphi,x,y,p)\leq\frac{1}{t_i}\varepsilon_{t_i}(\varphi^N,x,y,s).$$

Noting that $\lim_{i\to\infty}\frac{1}{t_i}\varepsilon_{t_i}(\varphi^N,x,y,s)=0$, we have

$$\lim_{i\to\infty}\frac{1}{m_i}\varepsilon_{m_i}(\varphi,x,y,p)=0.$$

This shows that

$$\underline{F}(\varphi,x,y,p)=0.$$

(2) Suppose $\overline{F}(\varphi^N,x,y,s)=1$ for all $s>0$. Fix $p>0$. Since the map $\varphi:[0,N]\times X\to X$ is uniform continuous, then exists $s>0$ such that for $\mu,\nu\in X$ and each $t\in[0,N]$, $d(\varphi(t,\mu),\varphi(t,\nu))<p$ provided $d(\mu,\nu)<s$. For such an s, $\overline{F}(\varphi^N,x,y,s)=1$, so there exists an increasing sequence $\{t_i\}$ such than for $i\to\infty$,

$$\lim_{i\to\infty}\frac{1}{t_i}\varepsilon_{t_i}(\varphi^N,x,y,s)=1.$$

Put $m_i=Nt_i$. We can see that

$$N\varepsilon_{t_i}(\varphi^N,x,y,s)\leq\varepsilon_{m_i}(\varphi,x,y,p).$$

then

$$\frac{1}{t_i}\varepsilon_{t_i}(\varphi^N,x,y,s)\leq\frac{1}{m_i}\varepsilon_{m_i}(\varphi,x,y,p).$$

For $i\to\infty, \lim_{i\to\infty}\frac{1}{t_i}\varepsilon_{t_i}(\varphi^N,x,y,s)=1$, then further

$$\lim_{i\to\infty}\frac{1}{m_i}\varepsilon_{m_i}(\varphi,x,y,p)=1.$$

Therefore, for all $p>0$,

$$\overline{F}(\varphi,x,y,p)=1.$$

Lemma 3.3 Let $\varphi:\mathcal{R}\times X\to X$ be a continuous flow, $x,y\in X$, $N>0$. If (x,y) is a Li-Yorke pair of φ, then (x,y) is a Li-Yorke pair of φ^N.

Proof If (x,y) is a Li-Yorke pair of φ, we have $\liminf_{t\to\infty} d(\varphi(t,x),\varphi(t,y))=0$ and $\limsup_{t\to\infty} d(\varphi(t,x),\varphi(t,y))>0.$

Then there are two infinite sequences $\{s_i\}$, $\{t_i\}$ of $\mathbb{R}$ such that

$$\lim_{i\to\infty} d(\varphi(s_i,x),\varphi(s_i,y))=0 \text{ and } \lim_{i\to\infty} d(\varphi(t_i,x),\varphi(t_i,y))>0.$$

Put $s_i=\frac{s_i}{N}$ *and* $t_i=\frac{t_i}{N}$. Hence, for $i\to\infty$, we have

$$\lim_{i\to\infty} d(\varphi^N(s_{i'},x),\varphi^N(s_{i'},y))=\lim_{i\to\infty} d(\varphi(s_i,x),\varphi(s_i,y))=0.$$

$$\lim_{i\to\infty} d(\varphi^N(t_{i'},x),\varphi^N(t_{i'},y))=\lim_{i\to\infty} d(\varphi(t_i,x),\varphi(t_i,y))>0.$$

This shows that (x,y) is a Li-Yorke pair of φ^N.

Lemma 3.4 Let $\varphi:\mathcal{R}\times X\to X$ be a continuous flow, $x,y\in X$, $N>0$. If (x,y) is a Li-Yorke pair of φ^N,then (x,y) is a Li-Yorke pair of φ.

Proof If (x,y) is a Li-Yorke pair of φ^N, that is

$$\liminf_{t\to\infty} d(\varphi^N(t,x),\varphi^N(t,y))=0 \quad and \quad \limsup_{t\to\infty} d(\varphi^N(t,x),\varphi^N(t,y))>0.$$

Then there are two infinite sequences $\{s_i\}$, $\{t_i\}$ of $\mathbb{R}$ such that

$$\lim_{i\to\infty} d(\varphi^N(s_i,x),\varphi^N(s_i,y))=0 \quad and \quad \lim_{i\to\infty} d(\varphi^N(t_i,x),\varphi^N(t_i,y))>0.$$

Put $s_{i'}=s_iN$ *and* $t_{i'}=t_iN$. Therefore, when $i\to\infty$,

$$\lim_{i\to\infty} d(\varphi(s_{i'},x),\varphi(s_{i'},y))=\lim_{i\to\infty} d(\varphi^N(s_i,x),\varphi^N(s_i,y))=0.$$

$$\lim_{i\to\infty} d(\varphi(t_{i'},x),\varphi(t_{i'},y))=\lim_{i\to\infty} d(\varphi^N(t_i,x),\varphi^N(t_i,y))>0.$$

Thus (x,y) is a Li-Yorke pair of $N>0$.

4. Main results and proofs

Theorem 4.1 Let (X,d) be a compact metric space , $\varphi:\mathcal{R}\times X\to X$ be a continuous flow, $N>0$ an integer. Then φ is distributional chaotic if and only if φ^N is distributional chaotic.

Proof By Lemma 3.1 and Lemma 3.2 we knows that for any $N>0$, φ is distributional chaotic if and only if φ^N is distributional chaotic.

Theorem 4.2 Let (X,d) be a compact metric space , $\varphi:\mathcal{R}\times X\to X$ be a continuous flow, $N>0$ an integer. Then φ is Li-Yorke chaotic if and only if φ^N is Li-Yorke chaotic.

Proof By Lemma 3.3 and Lemma 3.4 we knows that for any $N>0$, φ is Li-Yorke chaotic if and only if φ^N is Li-Yorke chaotic.

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