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Characterization Results based on Quantile version of Two Parametric Generalized Entropy of Order Statistics

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Abstract: The Quantile based entropy measure own a few precise properties than its distribution function technique. In this article, the idea of quantile based Bilal's and Baig's uncertainty measure is prolonged for order statistics for residual and past lifetimes and have a look at their properties, this two parametric entropy measure characterizes the distribution characteristic uniquely. A few characterization results of generalized residual entropy of order statistics and monotonicity property is likewise mentioned.

1. Introduction

An outstanding and significant idea in the field of information theory is the estimation of uncertainty or for the most part known as Shannon's entropy was initially dictated by Claude Shannon in [20]. These days, there is an incredible attraction of this measure among the scientists to generally apply it in various sciences like material science, computer sciences, probability, communication theory and reliability. let *W* be a random variable (r.v.) having an absolutely continuous cdf *F(w)* and pdf *f(w)*, then the basic uncertainty measure of *W* is defined as

$$H_f(W) = H(f) = -\int_0^{\infty} f(w)\log f(w)dw$$

(1)

H(w) is commonly referred as to the Shannon information measure of *W* or Shannon entropy. There is now a huge literature devoted to the applications, generalizations and properties of Shannon measure of uncertainty refer to Ref. [1]. A two parametric extension of Shannon entropy measure is given by Bilal and Baig ref to Ref. [3] is given as

$$H^{a,b}(W) = \frac{1}{a(b-a)}\left(\int_0^{\infty} f^{2-\frac{b}{a}}(w)dw - 1\right) \qquad \frac{b}{2} < a < b,\ b \geq 1$$

(2)

Particular cases,

Put $a = 1$, the measure (2) reduces to Mathai-Haubold when $b \to 1$, the measure (2) reduces to Shannon entropy as given in (1)

All of the theoretical investigations and applications the use of these facts measures are based totally on the distribution feature, however might not be suitable in conditions in which the distribution isn't always analytically tractable. an alternative technique to the examine is to apply the Quantile functions (QFS) which are given as follows

$$\wp(y) = F^{-1}(y) = \inf\{w \mid F(w) \ge y\} \qquad 0 \le y \le 1$$

(3)

The Quantile functions utilized in carried out works inclusive of numerous types of lambda distribution Van Staden and Loots [25], the Power-Pareto distribution Hankin and lee [12], Govindrajulu distribution do not have tractable distribution functions. New models and characterizations that are unresolvable within the distribution function approach can be resolved with the help of quantile feature we talk over with to Gilchrist [10]. Nair and Sankaran [18] and Nair et. al [19].

Sunoj and Sankaran [22] have considered the Quantile version of Shannon entropy and its residual form which are defined as

$$H = \int_0^1 \log \rho(x) dx$$

(4)

and

$$H(y) = H(W; \wp(y)) = \log(1-y) + (1-y)^{-1} \int_y^1 \log \rho(x) dx$$

(5)

respectively where $\rho(x) = \frac{d}{dy}\wp(y)$ is the Quantile density function. Defining the density quantile function by $f\wp(y) = f(\wp(y))$ and the quantile density function by $\rho(x)$, we have

$$\rho(x) f(\wp(y)) = 1$$

(6)

A quantile based entropy of order statistics and its dynamic shape is proposed by way of sunoj et al. [21]. Recently, Kumar and Rekha [16] and khayal and Tripathy [13] studied the properties of quantile based dynamic Tsallis entropy and order statistics in image and signal processing. We try to extend the concept of quantile based Generalized entropy using order statistics and study it for some lifetime distributions.

2. Generalized Two Parametric Quantile Based Entropy (GTPQBE) of $W_{i:n}$:

Analogous to (2), a GTPE for the ith order statistics $W_{i:n}$ is defined as

$$H^{a,b}_{W_{i:n}} = \frac{1}{a(b-a)}\left(\int_0^\infty f_{i:n}^{\,2-\frac{b}{a}}(w) dw - 1\right) \qquad \frac{b}{2} < a < b,\ b \ge 1 \tag{7}$$

where $f_{i:n}(w)$ is the pdf of the i^{th} order statistics is given by

$$f_{i:n}(w) = \frac{(F(w))^{(i-1)} (1-F(w))^{(n-i)} f(w)}{B(i, n-i+1)}$$

here $B(i,n-i+1)$ is the beta function with parameters i and $(n-i+1)$ for details refer to Refs [2]. From (3), we have $F(\wp(y))=y$, then pdf of the ith order statistics becomes

$$f_{i:n}(y)=f_{i:n}(\wp(y))=\frac{1}{B(i,n-i+1)}(y)^{(i-1)}(1-y)^{(n-i)}\,f((\wp(y)),$$

$$=\frac{1}{B(i,n-i+1)}(y)^{(i-1)}(1-y)^{(n-i)}\,\frac{1}{\rho(y)}$$

$$=\frac{g_i(y)}{\rho(y)}$$

here $g_i(y)$ follow the pdf of beta distribution with unknown parameters i and $(n-i+1)$, now using (3), the corresponding GTPQBE of $W_{i:n}$ is defined as

$$H^{a,b}_{W_{i:n}}=H^{a,b}_{W_{i:n}}(\wp(y))=\frac{1}{a(b-a)}\left(\int_0^1 f_{i:n}{}^{2-\frac{b}{a}}(\wp(y))d\wp(y)-1\right)\quad \frac{b}{2}<a<b,\ b\geq 1\ i^{th}\text{ order statistics}$$

$$=\frac{1}{a(b-a)}\left(\int_0^1 (g_i(y))^{2-\frac{b}{a}}(\rho(y))^{\frac{b}{a}-1}\,dy-1\right) \qquad (8)$$

The measure (8) is GTPQBE for i^{th} order statistics, when $a=1$, (8) reduces to Methai-Haubold entropy

$$H^{b}_{W_{i:n}}=\frac{1}{(b-1)}\left(\int_0^1 (g_i(y))^{2-b}(\rho(y))^{b-1}\,dy-1\right). \qquad (9)$$

when $b\to 1$, (9) reduces to Shannon entropy of i^{th} order statistics studied by Sunoj et.al [21].

$$H_{W_{i:n}}=-\int_0^1 g_i(y)\log\left(\frac{g_i(y)}{\rho(y)}\right)dy\ .$$

Example 2.1: If *W* is a r.v. following the Govindrajulu's [11] distribution that does not have any closed form expressions for distribution and density function, the QFS and quantile density function are given respectively by

$$\wp(y)=r\{(s+1)y^{s}-s\,y^{(s+1)}\}\ \text{ and }\ \rho(y)=r\,s\,(s+1)(1-y)y^{s-1}\qquad 0\leq y\leq 1, r, s>0$$

Thus GTPQBE of i^{th} order statistics for Govindrajulu's distribution is given as

$$H^{a,b}_{W_{i:n}}=\frac{1}{a(b-a)}\left(\int_0^1 (g_i(y))^{2-\frac{b}{a}}(\rho(y))^{\frac{b}{a}-1}\,dy-1\right)$$

$$=\frac{1}{a(b-a)}\left(\frac{\int_0^1 (y)^{(i-1)\left(2-\frac{b}{a}\right)}(1-y)^{(n-i)\left(2-\frac{b}{a}\right)}}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}\left(r\,s(s+1)(1-y)y^{s-1}\right)^{\frac{b}{a}-1}dy-1\right)$$

$$= \frac{1}{a(b-a)}\left((rs(s+1))^{\frac{b}{a}-1} \frac{B\left(\left(2-\frac{b}{a}\right)i + s\left(\frac{b}{a}-1\right), \left(2-\frac{b}{a}\right)(n-i-1)+2\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}} - 1 \right)$$

Example 2.2: If W is a r.v. follows the uniform distribution over $(r\,s)$ $r<s$, then its QFS and quantile density function are given respectively by

$$\wp(y) = r + (s-r)y \text{ and } \rho(y) = (s-r)$$

Thus GTPQBE of i^{th} order statistics for uniform distribution is given as

$$H^{a,b}_{W_{i:n}} = \frac{1}{a(b-a)}\left(\int_0^1 (g_i(y))^{2-\frac{b}{a}} (\rho(y))^{\frac{b}{a}-1} dy - 1 \right)$$

$$= \frac{1}{a(b-a)}\left(\frac{\int_0^1 (y)^{(i-1)\left(2-\frac{b}{a}\right)} (1-y)^{(n-i)\left(2-\frac{b}{a}\right)}}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}} (s-r)^{\frac{b}{a}-1} dy - 1 \right)$$

$$= \frac{1}{a(b-a)}\left((s-r)^{\frac{b}{a}-1} \frac{B\left(\left(2-\frac{b}{a}\right)(i-1)+1, \left(2-\frac{b}{a}\right)(n-i)+1\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}} - 1 \right)$$

3. Generalized Two Parametric Quantile Based Divergence Measure:

Divergence measure plays an important role in measuring the distance between two probability distribution functions. Let X and Y be two non-negative r.vs. With density functions f and g and survival functions $\bar{F}$ and $\bar{G}$ respectively. Then $\frac{g}{f}$ could be considered as a measure divergence of g from f in the sense that when $\frac{g}{f}$ is far from unity, the difference in the distributions is large. One can define a divergence measure of g from f as

$$H^{\varphi}(X:Y) = \int_0^{\infty} f(x)\varphi\left(\frac{f(x)}{g(x)}\right) dx$$

(10)

For different choice of φ in (10), we get the following divergence measure of g from f.

$$H^{a,b}(X:Y) = \frac{1}{a(b-a)}\left(\int_0^{\infty} f(x)\left(\frac{f(x)}{g(x)}\right)^{2-\frac{b}{a}} dx - 1 \right), \quad \frac{b}{2} < a < b,\ b \geq 1$$

(11)

which is the GTPQB (divergence) measure or relative entropy of g with respect to f. when $a=1$, then (11) reduces to Methai-Haubold divergence measure g from f.

$$H^{a,b}(X:Y)=\frac{1}{(b-1)}\left(\int_0^{\infty} f(x)\left(\frac{f(x)}{g(x)}\right)^{2-b} d\,x-1\right), \qquad b\geq 1$$

(12)

when $b\to 1$, (12) reduces to Kullback-Leibler relative entropy function.

Some goodness of fit test established based on entropy and divergence measures. Ebrahimi et al. [9] studied Kullback-Leibler [14] relative information measure and its properties for order statistics using Shannon entropy for order statistics.

Lemma 2.1: The GTPQBE between the distribution of order statistics $f_{i:n}$ and the original distribution *f* is given by

$$H^{a,b}(W_{i:n}:X)=-H^{a,b}(Y_i)$$

where Y_i is the beta distribution with parameters i and $(n-i+1)$.

Proof: From (11), we have

$$H^{a,b}(W_{i:n}:Y)=\frac{1}{a(b-a)}\left(\int_0^{\infty} f_{i:n}(w)\left(\frac{f_{i:n}(w)}{f(w)}\right)^{1-\frac{b}{a}} d\,w-1\right)$$

substituting the value of $f_{i:n}(w)$ in the above expression and after simplification, we get

$$H^{a,b}(W_{i:n}:Y)=\frac{1}{a(b-a)}\left(\int_0^{\infty} f_{i:n}(w)\frac{(\wp(y))^{\left(1-\frac{b}{a}\right)(i-1)}(1-\wp(y))^{\left(1-\frac{b}{a}\right)(n-i)}}{(B(i,n-i+1))^{1-\frac{b}{a}}} d\,w-1\right)$$

$$=\frac{1}{a(b-a)}\left(\int_0^{\infty} f_{i:n}(w)\frac{(w)^{\left(1-\frac{b}{a}\right)(i-1)}(1-w)^{\left(1-\frac{b}{a}\right)(n-i)}}{(B(i,n-i+1))^{1-\frac{b}{a}}} d\,w-1\right)$$

$$=\frac{1}{a(b-a)}\left(\frac{B\left(\left(1-\frac{b}{a}\right)(i-1)+1,\left(1-\frac{b}{a}\right)(n-i)+1\right)}{(B(i,n-i+1))^{1-\frac{b}{a}}}-1\right)$$

$$=-H^{a,b}(Y_i)$$

(13)

Hence, the GTPQBE between the distribution of order statistics and the original distribution is distribution free.

For some specific univariate lifetime distributions, the expression (8) is evaluated as given below in table 1.

Table 1: GTPQBE of $H^{a,b}_{W_{i:n}}$ of i^{th} order statistics for different lifetime Distributions

Distribution	**Quantile function** $\wp(y)$	$H^{a,b}_{W_{i:n}}$
Exponential	$-\theta^{-1}\ln(1-y)$	$\frac{1}{a(b-a)}\left((\theta)^{1-\frac{b}{a}}\frac{B\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i+1)-1\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Finite Range	$s\left\{1-(1-y)^{\frac{1}{r}}\right\}$	$\frac{1}{a(b-a)}\left(\left(\frac{s}{r}\right)^{\frac{b}{a}-1}\frac{B\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i+1)+1\right)+\left(\frac{b}{a}-1\right)\frac{1}{r}}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Generalized Pareto	$\frac{r}{s}\left[(1-y)^{-\left(\frac{r}{r+1}\right)}-1\right]$	$\frac{1}{a(b-a)}\left(\left(\frac{s}{r+1}\right)^{\frac{b}{a}-1}\frac{B\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i+1)+\left(2-\frac{b}{a}\right)\left(\frac{2r+1}{r+1}\right)+1\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Log-Logestic	$\frac{1}{r}\left(\frac{y}{1-y}\right)^{-\frac{1}{s}}$	$\frac{1}{a(b-a)}\left((rs)^{1-\frac{b}{a}}\frac{B\left(\left(2-\frac{b}{a}\right)i+\frac{1}{s}\left(1-\frac{b}{a}\right),\left(2-\frac{b}{a}\right)(n-i+1)+\frac{1}{s}\left(1-\frac{b}{a}\right)\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Pareto-I	$s(1-y)^{-\frac{1}{r}}$	$\frac{1}{a(b-a)}\left(\left(\frac{s}{r}\right)^{\frac{b}{a}-1}\frac{B\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i)+1\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Power Distribution	$r(y)^{\frac{1}{s}}$	$\frac{1}{a(b-a)}\left(\left(\frac{r}{s}\right)^{\frac{b}{a}-1}\frac{B\left(\left(2-\frac{b}{a}\right)i+\frac{1}{s}\left(\frac{b}{a}-1\right),\left(2-\frac{b}{a}\right)(n-i)+1\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Uniform	$r+(s-r)y$	$\frac{1}{a(b-a)}\left((s-r)^{\frac{b}{a}-1}\frac{B\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i)+1\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$

4. Generalized Two Parametric Quantile Based Residual Entropy (GTPQBRE):

In the context of reliability and life testing studies when the present age of a component needs to be incorporated, then the measure (1) and (2) are not appropriate. In this situation the residual lifetime of the system when it is still operating at time *t*, is $(W_t = W-t\,|\,W>t)$ which has the probability density $f(w:t)=\frac{f(w)}{\bar{F}(t)},\quad w\geq t>0.$ The residual entropy was considered by Ebrahimi [8], which basically measures the expected uncertainty contained in remaining lifetime of a system and is defined as

$$f(w:t)=\int_t^{\infty}\frac{f(w)}{\bar{F}(t)}\log\left(\frac{f(w)}{\bar{F}(t)}\right)dw,\quad w\geq t>0.$$

(14)

In analogous to Ebrahimi [8] refer to Refs. [15,17], GTPQBRE of r.v. *W* is given by

$$H^{a,b}(W:t)=\frac{1}{a(b-a)}\left(\int_t^{\infty}\left(\frac{f(w)}{\bar{F}(t)}\right)^{2-\frac{b}{a}}dw-1\right),\quad \frac{b}{2}<a<b,\ b\geq 1$$

(15)

Obviously when $t=0$ (14) and (15) reduces to entropy measures (1) and (2) respectively. Ref to ref. [24]

$$H^{a,b}(W_{i:n}:t)=\frac{1}{a(b-a)}\left(\int_t^{\infty}\left(\frac{f_{i:n}(w)}{\overline{F}_{i:n}(t)}\right)^{2-\frac{b}{a}}d\,w-1\right),\quad t\geq 0$$

(16)

$\overline{F}_{i:n}(W)=\frac{\overline{B}_{F(w)}(i,n-i+1)}{B(i,n-i+1)}$ is the survival function of the i^{th} order statistics and

$\overline{B}_w(u,v)=\int_x^1 w^{u-1}(1-w)^{v-1}d\,w,\ 0<w<1$ is the incomplete beta function, refer to Refs. [4,5,6].

The GTPQBRE of i^{th} order statistics is defined as

$$H^{a,b}_{W_{i:n}}(y)=H^{a,b}(W_{i:n}:\wp(y))=\frac{1}{a\,(b-a)}\left(\left(\frac{B(i,n-i+1)}{\overline{B}_w(i,n-i+1)}\right)^{2-\frac{b}{a}}\int_y^1 (g_i(y))^{2-\frac{b}{a}}(\rho(y))^{\frac{b}{a}-1}d\,y-1\right)\quad(17)$$

where $\frac{B(i,n-i+1)}{\overline{B}_w(i,n-i+1)}$ is the quantile form of survival function $\overline{F}_{i:n}(W)$. An equivalent representation of (17) is of the form of

$$a(b-a)H^{a,b}_{W_{i:n}}(y)=\frac{1}{\left(\overline{B}_w(i,n-i+1)\right)^{2-\frac{b}{a}}}\int_y^1 y^{\left(2-\frac{b}{a}\right)(i-1)}(1-y)^{\left(2-\frac{b}{a}\right)(n-i)}\rho(y)^{\frac{b}{a}-1}d\,y-1$$

which is rewritten as

$$a(b-a)\left(\overline{B}_y(i,n-i+1)\right)^{2-\frac{b}{a}}H^{a,b}_{W_{i:n}}(y)=\int_y^1 y^{\left(2-\frac{b}{a}\right)(i-1)}(1-y)^{\left(2-\frac{b}{a}\right)(n-i)}\rho(y)^{\frac{b}{a}-1}d\,y-\left(\overline{B}_y(i,n-i+1)\right)^{2-\frac{b}{a}}\quad(18)$$

Differentiating (18) w.r.t to 'y', we obtain

$$a(b-a)\left\{\left(\overline{B}_y(i,n-i+1)\right)^{2-\frac{b}{a}}\left(H^{a,b}_{W_{i:n}}(y)\right)'-\left(2-\frac{b}{a}\right)\left(\overline{B}_y(i,n-i+1)\right)^{1-\frac{b}{a}}y^{i-1}(1-y)^{n-i}H^{a,b}_{W_{i:n}}(y)\right\}$$

$$=\left(2-\frac{b}{a}\right)\left(\overline{B}_y(i,n-i+1)\right)^{1-\frac{b}{a}}y^{i-1}(1-y)^{n-i}-y^{(i-1)\left(2-\frac{b}{a}\right)}(1-y)^{(n-i)\left(2-\frac{b}{a}\right)}(q(y))^{\frac{b}{a}-1}$$

It can be rewritten as

$$y^{(i-1)\left(2-\frac{b}{a}\right)}(1-y)^{(n-i)\left(2-\frac{b}{a}\right)}(q(y))^{\frac{b}{a}-1}=\left(2-\frac{b}{a}\right)\left(\overline{B}_y(i,n-i+1)\right)^{1-\frac{b}{a}}y^{i-1}(1-y)^{n-i}-a(b-a)\left(\overline{B}_y(i,n-i+1)\right)^{2-\frac{b}{a}}\left(H^{a,b}_{W_{i:n}}(y)\right)'$$

$$+a(b-a)\left(2-\frac{b}{a}\right)\left(\overline{B}_y(i,n-i+1)\right)^{1-\frac{b}{a}}y^{i-1}(1-y)^{n-i}H^{a,b}_{W_{i:n}}(y)$$

This gives

$$(q(y))^{\frac{b}{a}-1}=\frac{\left(\overline{B}_y(i,n-i+1)\right)^{1-\frac{b}{a}}y^{i-1}(1-y)^{n-i}}{y^{(i-1)\left(2-\frac{b}{a}\right)}(1-y)^{(n-i)\left(2-\frac{b}{a}\right)}}\left(\left(2-\frac{b}{a}\right)+a(b-a)\left(2-\frac{b}{a}\right)H^{a,b}_{W_{i:n}}(y)\right)$$

$$-\frac{a(b-a)\left(\overline{B}_y(i,n-i+1)\right)^{2-\frac{b}{a}}\left(H^{a,b}_{W_{i:n}}(y)\right)'}{y^{(i-1)\left(2-\frac{b}{a}\right)}(1-y)^{(n-i)\left(2-\frac{b}{a}\right)}}$$

(19)

Equation (19) gives an immediate connection between the quantile density function $q(y)$ and $H^{a,b}_{W_{i:n}}(y)$ which show that the quantile based generalized two parametric residual entropy of i^{th} order statistics $H^{a,b}_{W_{i:n}}(y)$ uniquely decides the underlying distribution.

Remark 3.1: when $a=1,\ b\to 1$**,** then (17) reduces to

$$H^{a,b}_{W_{i:n}}(y)=-\frac{B(i,n-i+1)}{\overline{B}_y(i,n-i+1)}\int_y^1 g_i(y)\log(g_i(y))d\,y+\frac{B(i,n-i+1)}{\overline{B}_y(i,n-i+1)}\int_y^1 g_i(y)\log(\rho(y))d\,y+\log\left(\frac{\overline{B}_y(i,n-i+1)}{B(i,n-i+1)}\right)$$

A quantile version of Shannon residual entropy of i^{th} order statistics obtained by Sunoj et. al, [21].

Example 3.1: Let $\overline{W}_i$ be the i^{th} order statistics based on a random sample of size *n* from uniform distribution on (0,1). Then

$$H^{a,b}_{W_{i:n}}(y)=\frac{1}{a(b-a)}\left((s-r)^{\frac{b}{a}-1}\frac{B\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i)+1\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$$

Example 3.2: Let $\overline{W}_i$ be the i^{th} order statistics based on a random sample of size *n* from standard exponential distribution on (0,1). Then

$$H^{a,b}_{W_{i:n}}(y)=\frac{1}{a(b-a)}\left(\frac{B\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i+1)\right)}{B(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$$

Example 3.3: Let $\overline{W}_i$ be the i^{th} order statistics based on a random sample of size *n* from Pareto distribution with quantile function

$$\wp(y)=s(1-y)^{-\frac{1}{r}}\ \text{and}\ \rho(y)=\frac{s}{r}(1-y)^{-\left(\frac{1}{r}+1\right)}$$

and for computing $H^{a,b}_{W_{i:n}}(y)$, we have $(\rho(y))^{\frac{b}{a}-1}=\left(\frac{s}{r}(1-y)^{-\left(\frac{1}{r}+1\right)}\right)^{\frac{b}{a}-1}$ .Thus the GTPQBRE (17) of the i^{th} order statistics for Pareto distribution is given as

$$H^{a,b}_{W_{i:n}}(y)==\frac{1}{a(b-a)}\left(\frac{1}{\left(\overline{B}_y(i,n-i+1)\right)^{2-\frac{b}{a}}}\int_y^1 (y)^{\left(2-\frac{b}{a}\right)(i-1)}(1-y)^{\left(2-\frac{b}{a}\right)(n-i)}\left(\frac{s}{r}(1-y)^{-\left(\frac{1}{r}+1\right)}\right)^{\frac{b}{a}-1}d\,y-1\right)$$

is gives

$$H_{W_{i:n}}^{a,b}(y)=\frac{1}{a,(b-a)}\left(\left(\frac{s}{r}\right)^{\frac{b}{a}-1}\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i+1)+\left(1-\frac{b}{a}\right)\frac{1}{r}-1\right)}{\overline{B}_y(i,n-i+1)^{2-\frac{b}{a}}}-1\right)$$

Table 2: GTPQBRE of $H_{W_{i:n}}^{a,b}$ of i^{th} order statistics for different lifetime Distributions

Distribution	**Quantile function** $\wp(y)$	$H_{W_{i:n}}^{a,b}(y)$
Exponential	$-\theta^{-1}\ln(1-y)$	$\frac{1}{a(b-a)}\left(\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i+1)-1\right)}{\overline{B}_y(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Finite Range	$s\left\{1-(1-y)^{\frac{1}{r}}\right\}$	$\frac{1}{a(b-a)}\left(\left(\frac{s}{r}\right)^{\frac{b}{a}-1}\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i+1)+1\right)+\left(1-\frac{b}{a}\right)\frac{1}{r}}{\overline{B}_y(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Generalized Pareto	$\frac{r}{s}\left[(1-y)^{-\left(\frac{r}{r+1}\right)}-1\right]$	$\frac{1}{a(b-a)}\left(\left(\frac{s}{r+1}\right)^{\frac{b}{a}-1}\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i)+\left(1-\frac{b}{a}\right)\left(\frac{2r+1}{r+1}\right)+1\right)}{\overline{B}_y(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Log-Logestic	$\frac{1}{r}\left(\frac{y}{1-y}\right)^{-\frac{1}{s}}$	$\frac{1}{a(b-a)}\left((rs)^{1-\frac{b}{a}}\frac{B\left(\left(2-\frac{b}{a}\right)i+\frac{1}{r}-\frac{3}{s},\left(2-\frac{b}{a}\right)(n-i+1)-\frac{1}{r}+\frac{1}{s}\right)}{\overline{B}_y(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Pareto-I	$s(1-y)^{-\frac{1}{r}}$	$\frac{1}{a(b-a)}\left(\left(\frac{s}{r}\right)^{\frac{b}{a}-1}\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i+1)+\left(1-\frac{b}{a}\right)\frac{1}{r}\right)}{\overline{B}_y(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Power Distribution	$r(y)^{\frac{1}{s}}$	$\frac{1}{a(b-a)}\left(\left(\frac{r}{s}\right)^{\frac{b}{a}-1}\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)i+\frac{1}{r}-\frac{1}{s},\left(2-\frac{b}{a}\right)(n-i)+1\right)}{\overline{B}_y(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$
Uniform	$r+(s-r)y$	$\frac{1}{a(b-a)}\left(\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)(i-1)+1,\left(2-\frac{b}{a}\right)(n-i)+1\right)}{\overline{B}_y(i,n-i+1)^{\left(2-\frac{b}{a}\right)}}-1\right)$

5. Characterization of Generalized two parametric Quantile Based Entropy On the First Order Statistics:

Next, we acquire the characterization result dependent on GTPQBE of the first order statistics. Let $W_{1:n}$ and $W_{n:n}$ be the first order statistics and last order statistic in a random sample $\{W_1,W_2,...,W_n\}$ of size n from a positive and continuous random variable W with cdf F and pdf f, then the distribution function, the density function and the hazard rate function of first order statistics $W_{1:n}$ are respectively given by

$$F_{1:n}(w)=1-\overline{F}^{\,n}(w).$$

$$f_{1:n}(w)=n\overline{F}^{\,n-1}(w)\,f(w),$$

and

$$K_{1:n}(w)=\frac{f_{1:n}(w)}{\overline{F}_{1:n}(w)}=\frac{n\,f(w)}{\overline{F}(w)}.$$

putting $i=1$ in (17), we get the GTPQBE of first order statistics, given as

$$H^{a,b}_{W_{1:n}}(y)=\frac{1}{a(b-a)}\left(\left(\frac{B(1,n)}{\overline{B}_y(1,n)}\right)^{2-\frac{b}{a}}\int_y^1 (g_1(y))^{2-\frac{b}{a}}(\rho(y))^{\frac{b}{a}-1}dy-1\right),$$

$$=\frac{1}{a(b-a)}\left(\left(\frac{n^{2-\frac{b}{a}}}{(1-y)^{n\left(2-\frac{b}{a}\right)}}\right)\int_y^1 (1-y)^{\left(2-\frac{b}{a}\right)(n-1)}(\rho(y))^{\frac{b}{a}-1}dy-1\right)$$

(20)

The GTPQBE for sample maximum order statistics $W_{n:n}$ can be obtained from (17) by taking $i=n$, as

$$H^{a,b}_{W_{n:n}}(y)=\frac{1}{a(b-a)}\left(\left(\frac{B(n,1)}{\overline{B}_y(n,1)}\right)^{2-\frac{b}{a}}\int_y^1 (g_n(y))^{2-\frac{b}{a}}(\rho(y))^{\frac{b}{a}-1}dy-1\right),$$

$$=\frac{1}{a(b-a)}\left(\left(\frac{n}{(1-y^n)}\right)^{2-\frac{b}{a}}\int_y^1 (y)^{\left(2-\frac{b}{a}\right)(n-1)}(\rho(y))^{\frac{b}{a}-1}dy-1\right)$$

(21)

A natural question emerges that whether the GTPQBE $H^{a,b}_{W_{1:n}}(y)$ decides the lifetime distribution $F(.)$ uniquely. In the following example, we show this property.

Example 4.1: If *W* is a random variable following the Govindarajulu's distribution with quantile function given as

$$\wp(y)=r\left\{(s+1)y^s-s\,y^{s+1}\right\},\quad 0\le y\le 1;\ r,s>0 \text{ and } \quad \rho(y)=r\,s\,(s+1)\,y^{(s-1)}\,(1-y)$$

then the GTPQBRE of sample minimum for Govinarajulu's distribution is given as

$$H^{a,b}_{W_{1:n}}(y)=\frac{1}{a(b-a)}\left(\frac{1}{\left(\overline{B}_y(1,n)\right)^{2-\frac{b}{a}}}\int_y^1 (1-y)^{\left(2-\frac{b}{a}\right)(n-1)}\left(r\,s\,(s+1)\,y^{(s-1)}(1-y)\right)^{\frac{b}{a}-1}dy-1\right)$$

$$=\frac{1}{a(b-a)}\left(\frac{(rs(s+1))^{\frac{b}{a}-1}}{\left(\overline{B}_y(1,n)\right)^{2-\frac{b}{a}}}\overline{B}_y\left(\left(\frac{b}{a}-1\right)(s-1)+1,\left(2-\frac{b}{a}\right)(n-1)+\frac{b}{a}\right)-1\right)$$

Example 4.2: Let *W* be a random variable having the exponential distribution with quantile density function $-\theta^{-1}(1-y)^{-1},\quad \theta>0$. For series system $i=1$, we obtain

$$H^{a,b}_{W_{1:n}}(y)=\frac{1}{a(b-a)}\left(\frac{(n\theta)^{1-\frac{b}{a}}}{\left(2-\frac{b}{a}\right)}-1\right)$$

On the other hand, we have $H_W^{a,b}(y) = \frac{1}{a(b-a)}\left(\frac{(\theta)^{1-\frac{b}{a}}}{\left(2-\frac{b}{a}\right)} - 1\right)$

This give $H_{W_{1:n}}^{a,b}(y) - H_W^{a,b}(y) = \frac{(\theta)^{1-\frac{b}{a}}}{a(b-a)\left(2-\frac{b}{a}\right)}\left(n^{1-\frac{b}{a}} - 1\right)$

So, in the exponential case the difference between GTPQBRE of the life time of a series system and GTPQBRE of lifetime of each component is independent of y and depends only on parameters (a,b) and number of components of the system.

An important quantile measure useful in reliability analysis is the hazard quantile function, which is correspondent to the well-known hazard rate function $K(w) = \frac{f(w)}{\bar{F}(w)} = \frac{f(w)}{1-F(w)}$, which is defined as

$$Q(w) = K(\wp(y)) = \frac{f(\wp(y))}{(1-y)} = \frac{1}{(1-y)\rho(y)}$$

(22)

Next, we express a characterization result GTPQBRE of the first order statistics. By considering a connection between GTPQBRE $H_{W_{1:n}}^{a,b}(y)$ and the hazard quantile function $Q_{1:n}(x)$ of the first order statistics.

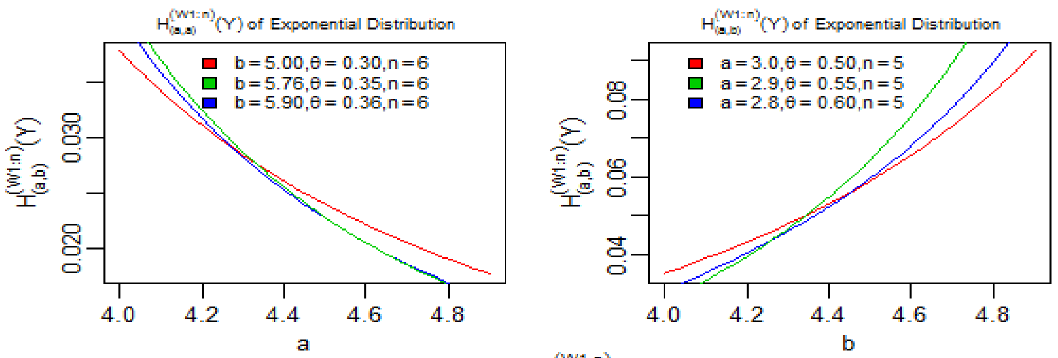

Fig.1. Graph of $H_{(a,b)}^{(W1:n)}(Y)$ for Parameters a and b

It is clear from Fig.1 that $H_{W_{1:n}}^{a,b}(y)$ is monotonically decreasing with respect to a but increasing with respect to b.

Theorem 4.1: Let $W_{1:n}$ denote the first order statistics with survival function $\bar{F}_{1:n}(w)$ and hazard quantile function $Q_{W_{1:n}}(y)$. Then the GTPQBRE $H_{W_{1:n}}^{a,b}(y)$ is given by

$$a(b-a)H_{W_{1:n}}^{a,b}(y) = \left\{C\left(Q_{W_{1:n}}(y)\right)^{1-\frac{b}{a}} - 1\right\}, \quad \frac{2}{b} < a < b, \quad b \geq 1 \tag{23}$$

if and only if;

a) For $C = \frac{1}{\left(2-\frac{b}{a}\right)}$, W has exponential distribution;

b) For $C < \frac{1}{\left(2-\frac{b}{a}\right)}$, W has Pareto distribution;

c) For $C > \frac{1}{\left(2-\frac{b}{a}\right)}$, W has Finite range distribution;

Proof: Let (23) holds, then

$$\left(\left(\frac{n^{2-\frac{b}{a}}}{(1-y)^{n\left(2-\frac{b}{a}\right)}}\right)\int_y^1 (1-y)^{\left(2-\frac{b}{a}\right)(n-1)}(\rho(y))^{\frac{b}{a}-1}d\,y-1\right)=\left\{C\left(Q_{W_{1:n}}(y)\right)^{1-\frac{b}{a}}-1\right\}$$

The quantile function of exponential distribution is $\wp(y)=-\theta^{-1}\ln(1-y)$ and $\rho(y)=\theta^{-1}(1-y)^{-1}$

substituting $Q_{W_{1:n}}(y)=\frac{f_{1:n}(\wp(y))}{(1-F(\wp(y)))^n}=\frac{n}{(1-y)\rho(y)}$ and simplifying, it gives

$$n\int_y^1 (1-y)^{\left(2-\frac{b}{a}\right)(n-1)}(\rho(y))^{\frac{b}{a}-1}d\,y=C\,(1-y)^{n\left(2-\frac{b}{a}\right)+\frac{b}{a}-1}\,(\rho(y))^{1-\frac{b}{a}}$$

differentiating above expression on both sides w.r.t ‘y’, we get after some simplifications

$$\frac{\rho'(y)}{\rho(y)}=\frac{1}{(1-y)}\left(\frac{n\left(2C-\frac{b}{a}C-1\right)}{C\left(\frac{b}{a}-1\right)}+1\right)$$

To solve this differential equation, we integrate it on both sides w.r.t ‘y’, we get

$$\rho(y)=C_1\,(1-y)^{\left(\frac{n\left(2C-\frac{b}{a}C-1\right)}{C\left(1-\frac{b}{a}\right)}-1\right)}$$

where C_1 is constant. Thus the underlying distribution is exponential if $C=\frac{1}{2-\frac{b}{a}}$, Pareto distribution if $C<\frac{1}{2-\frac{b}{a}}$ and finite range distribution if $C>\frac{1}{2-\frac{b}{a}}$.

The GTPQBE for $i=1$, i.e, the lifetime of the series systems, for several well-known distributions are provided in Table 3

Table 3: GTPQBRE of $H_{W_{1:n}}^{a,b}$ of i[th] order statistics for different lifetime Distributions for a series system.

Distribution	**Quantile function** $\wp(y)$	$H_{W_{1:n}}^{a,b}(y)$
Exponential	$-\theta^{-1}\ln(1-y)$	$\frac{1}{a(b-a)}\left(\frac{(n\theta)^{1-\frac{b}{a}}}{\left(2-\frac{b}{a}\right)}-1\right)$
Finite Range	$s\left\{1-(1-y)^{\frac{1}{r}}\right\}$	$\frac{1}{a(b-a)}\left((nr)^{2-\frac{b}{a}}\frac{(1-y)^{\frac{b}{a}-1}}{\left(nr\left(2-\frac{b}{a}\right)-\left(2-\frac{b}{a}\right)+1\right)(s)^{\left(1-\frac{b}{a}\right)}}-1\right)$
Generalized Pareto	$\frac{r}{s}\left[(1-y)^{-\left(\frac{r}{r+1}\right)}-1\right]$	$\frac{1}{a(b-a)}\left(\left(\frac{s}{r+1}\right)^{\frac{b}{a}-1}\frac{\left(n\left(2-\frac{b}{a}+1\right)\right)^{\left(2-\frac{b}{a}\right)}(1-y)^{\frac{\left(1-\frac{b}{a}\right)r}{(r+1)}}}{\left(nr\left(2-\frac{b}{a}\right)+n\left(2-\frac{b}{a}\right)+r\left(2-\frac{b}{a}\right)-r\right)(s)^{\left(1-\frac{b}{a}\right)}}-1\right)$
Govindarajulu's	$r\left\{(s+1)y^{s}-s\,y^{s+1}\right\}$	$\frac{1}{a(b-a)}\left(\{(rs)(1+s)\}^{\frac{b}{a}-1}n^{\left(2-\frac{b}{a}\right)}\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)+s\left(\frac{b}{a}-1\right),\left(2-\frac{b}{a}\right)(n-2)+2\right)}{(1-y)^{n\left(2-\frac{b}{a}\right)}}-1\right)$
Log-Logestic	$\frac{1}{r}\left(\frac{y}{1-y}\right)^{-\frac{1}{s}}$	$\frac{1}{a(b-a)}\left((rs)^{1-\frac{b}{a}}\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)+\frac{1}{s}\left(\frac{b}{a}-1\right),\left(2-\frac{b}{a}\right)n+\left(1-\frac{b}{a}\right)\frac{1}{s}\right)}{(rs)^{\frac{b}{a}-1}(1-y)^{n\left(2-\frac{b}{a}\right)}}n^{\left(2-\frac{b}{a}\right)}-1\right.$
Pareto-I	$s(1-y)^{-\frac{1}{r}}$	$\frac{1}{a(b-a)}\left(\frac{(nr)^{\left(2-\frac{b}{a}\right)}(1-y)^{\left(1-\frac{b}{a}\right)\frac{1}{r}}}{\left(nr\left(2-\frac{b}{a}\right)-n\left(2-\frac{b}{a}\right)+r\left(2-\frac{b}{a}\right)-r\right)(s)^{\left(1-\frac{b}{a}\right)}}-1\right)$
Power Distribution	$r(y)^{\frac{1}{s}}$	$\frac{1}{a(b-a)}\left(\left(\frac{r}{s}\right)^{\frac{b}{a}-1}\frac{\overline{B}_y\left(\left(2-\frac{b}{a}\right)+\frac{\left(\frac{b}{a}-1\right)}{s},\left(2-\frac{b}{a}\right)(n-1)+1\right)n^{\left(2-\frac{b}{a}\right)}}{(1-y)^{n\left(2-\frac{b}{a}\right)}}-1\right.$
Uniform	$r+(s-r)y$	$\frac{1}{a(b-a)}\left(\frac{n^{\left(2-\frac{b}{a}\right)}\{(s-r)(1-y)\}^{\frac{b}{a}-1}}{n\left(2-\frac{b}{a}\right)-\left(2-\frac{b}{a}\right)+1}-1\right)$

6. Characterization of Generalized two parametric Quantile Based Entropy on the n[th] Order Statistics:

In many naturalistic situations, the random variables are not necessary identified with the future only, but they can also cite to past. Such a random variable can be called inactivity time ${}_tW=[t-W \mid W<t]$. for fixed $t\geq 0$, which gives the time elapsed from the failure of a component given that its failure is less than or equal to *t*. It is also called the reversed residual lifetime. The past lifetime r.v. ${}_tW$ is connected with the reversed hazard rate function characterized by

$$\overline{Q}(w)=\overline{K}(\wp(y))=\frac{f(\wp(y))}{F(\wp(y))}=(y\,\rho(y))^{-1} \tag{24}$$

The reversed hazard rate function is quite practicable in the forensic science. Where exact time of failure (eg. Death in case of human beings) of a unit is of interest. Based on this idea, Di Crescenzo and Longobardi [7] defined uncertainty related to the r.v. ${}_tW=[t-W \mid W<t]$ as

$$\overline{H}(w:t) = -\int_0^t \frac{f(w)}{F(t)} \log\left(\frac{f(w)}{F(t)}\right) dw, \quad w \ge t > 0.$$

Sunoj and Sankaran [23] have considered the quantile version of Shannon past entropy, which is defined as

$$\overline{H}(y) = \overline{H}(W; \wp(y)) = \log(1-y) + (1-y)^{-1} \int_0^y \log \rho(x) dx.$$

Analogous to the GTPE, its past entropy measure is given as

$$\overline{H}^{a,b}(W:t) = \frac{1}{a(b-a)}\left(\int_0^t \left(\frac{f(w)}{F(t)}\right)^{2-\frac{b}{a}} dw - 1\right), \quad \frac{b}{2} < a < b,\ b \ge 1$$

(25)

For more details, refer to Nanda and Paul [17]. Analogous to (25), GTPE of the i^{th} order statistics $W_{i:n}$ is defined as

$$\overline{H}^{a,b}(W_{i:n}:t) = \frac{1}{a(b-a)}\left(\int_0^t \left(\frac{f_{i:n}(w)}{F_{i:n}(t)}\right)^{2-\frac{b}{a}} dw - 1\right), \quad t \ge 0$$

$$= \frac{1}{a(b-a)}\left(\frac{\int_0^t f_{i:n}^{2-\frac{b}{a}}(w) dw}{\left(B_{F(y)}(i, n-i+1)\right)^{2-\frac{b}{a}}} - 1\right)$$

(26)

where $B_{F(y)}(i, n-i+1)$ is the distribution function of the i^{th} order statistics. In terms of quantile function (26) can be expressed as follows

$$\overline{H}^{a,b}_{W_{i:n}}(y) = \overline{H}^{a,b}(W_{i:n} : \wp(y)) = \frac{1}{a(b-a)}\left(\frac{1}{\left(B_{F(\wp(y))}(i, n-i+1)\right)^{2-\frac{b}{a}}} \int_0^y (f_{i:n}(\wp(y)))^{2-\frac{b}{a}} d(\wp(y)) - 1\right). \tag{27}$$

The measure (27) may be considered as the GTPE for the i^{th} order statistics $W_{i:n}$ of inactivity time. Last order statistics is the significant case of order statistics. For $i = n$, we have $f_{n:n}(w) = nF^{n-1}(w) f(w)$, and $F_{n:n}(w) = (F(w))^n$. Thus the GTPQBE for the nth order statistics $W_{n:n}$ is defined as

$$\overline{H}^{a,b}(W_{n:n} : \wp(y)) = \frac{1}{a(b-a)}\left(\frac{n^{2\left(2-\frac{b}{a}\right)}}{(y)^{n\left(2-\frac{b}{a}\right)}} \int_0^1 (y)^{\left(2-\frac{b}{a}\right)(n-1)} (\rho(y))^{\frac{b}{a}-1} dy - 1\right). \tag{28}$$

Let W be a Power series distribution with quantile function and quantile density function are given respectively by $\wp(y)=r\,y^{\frac{1}{s}}$ and $\rho(y)=\dfrac{r\,y^{\frac{1}{s}-1}}{s}$ $\quad r,s>0$.

Next we state a characterization result the GTPQBE of reversed residual hazard lifetime of parallel systems. In this context, we establish that the Power series distribution can be characterized in terms of $\overline{H}^{a,b}(W_{n:n}:\wp(y))$.

Theorem 5.1: Let $W_{n:n}$ denote the last order statistics with survival function $\overline{F}_{n:n}(w)$ and the reversed hazard quantile function $\overline{Q}_{W_{1:n}}(y)$ is expressed as

$$a(b-a)\overline{H}^{a,b}_{W_{1:n}}(y)=\left\{C\left(\overline{Q}_{w_{n:n}}(y)\right)^{1-\frac{b}{a}}-1\right\},\quad \frac{2}{b}<a<b,\ \ b\ge 1 \tag{29}$$

if and only if W has power distribution function.

Proof: The reversed hazard quantile function for sample maxima of Power distribution is $\overline{Q}_{W_{n:n}}(y)=\dfrac{f_{n:n}(\wp(y))}{F_{n:n}(\wp(y))}=\dfrac{n\,f(\wp(y))}{F(\wp(y))}=n(y\,\rho(y))^{-1}=\dfrac{n\,b\,y^{\frac{1}{s}}}{r}$.

Let (29) holds, then $\quad\dfrac{n^{2\left(2-\frac{b}{a}\right)}}{y^{n\left(2-\frac{b}{a}\right)}}\displaystyle\int_0^y (y)^{\left(2-\frac{b}{a}\right)(n-1)}(\rho(y))^{\frac{b}{a}-1}d\,y=C\left(\overline{Q}_{w_{n:n}}(y)\right)^{1-\frac{b}{a}}$

substituting $\overline{Q}_{W_{n:n}}(y)=n(y\,\rho(y))^{-1}$, gives

$$n^{2\left(2-\frac{b}{a}\right)}\int_0^y (y)^{\left(2-\frac{b}{a}\right)(n-1)}(\rho(y))^{\frac{b}{a}-1}d\,y=C\,y^{n\left(2-\frac{b}{a}\right)}\left(\overline{Q}_{w_{n:n}}(y)\right)^{1-\frac{b}{a}}$$

Differentiating both sides w.r.t ‘y’ and after some simplifications, this reduces to

$$\frac{\rho'(y)}{\rho(y)}=\left(\frac{n^{3-\frac{b}{a}}-C\left\{\left(2-\frac{b}{a}\right)(n-1)+1\right\}}{C\left(\frac{b}{a}-1\right)}\right)\frac{1}{y}$$

To solve this differential equation, integrate on both sides w.r.t ‘y’, we get

$$\rho(y)=C_1\,y^{\frac{n^{\left(2-\frac{b}{a}\right)+1}-C\left\{n\left(2-\frac{b}{a}\right)-\left(2-\frac{b}{a}\right)+1\right\}}{C-C\left(2-\frac{b}{a}\right)}}$$

where C_1 is a constant, which characterizes the Power distribution for $C=\left(\dfrac{s\,n^{\left(2-\frac{b}{a}\right)+1}}{n\ s\left(2-\frac{b}{a}\right)-\left(2-\frac{b}{a}\right)+1}\right)$

Remark 5.1: If $C=\left(\frac{n^{\left(2-\frac{b}{a}\right)+1}}{n\ s\left(2-\frac{b}{a}\right)-\left(2-\frac{b}{a}\right)+1}\right)$, then equation (29) is a characterization of uniform distribution.

Table 4: GTPQB past Entropy of $H^{a,b}_{W_{n:n}}$ of i^{th} order statistics for different lifetime Distributions for a Parallel systems.

Distribution	**Quantile function** $\wp(y)$	$\overline{H}^{a,b}_{W_{n:n}}(y)$
Exponential	$-\theta^{-1}\ln(1-y)$	$\frac{1}{a(b-a)}\left(\frac{n^{2\left(2-\frac{b}{a}\right)}\theta^{1-\frac{b}{a}}B_y\left(\left(2-\frac{b}{a}\right)\lvert(n-1)\rvert,\left(2-\frac{b}{a}\right)\right)}{y^{n\left(2-\frac{b}{a}\right)}}-1\right)$
Generalized Pareto	$\frac{r}{s}\left[(1-y)^{-\left(\frac{r}{r+1}\right)}-1\right]$	$\frac{1}{a(b-a)}\left(\frac{(s)^{\frac{b}{a}-1}n^{2\left(2-\frac{b}{a}\right)}B_y\left(\left(2-\frac{b}{a}\right)(n-1)+1,\left(1-\frac{b}{a}\right)\left(\frac{2r+1}{r+1}\right)+1\right)}{y^{n\left(2-\frac{b}{a}\right)}(r+1)^{\frac{b}{a}-1}}-1\right)$
Govindarajulu's	$r\left\{(s+1)y^{s}-s\,y^{s+1}\right\}$	$\frac{1}{a(b-a)}\left(\frac{(rs(s+1))^{\frac{b}{a}-1}n^{2\left(2-\frac{b}{a}\right)}B_y\left(n\left(2-\frac{b}{a}\right)+s\left(\frac{b}{a}-1\right),\frac{b}{a}\right)}{y^{n\left(2-\frac{b}{a}\right)}}-1\right)$
Power Distribution	$r(y)^{\frac{1}{s}}$	$\frac{1}{a(b-a)}\left(\frac{r^{\frac{b}{a}-1}s^{\left(2-\frac{b}{a}\right)}n^{2\left(2-\frac{b}{a}\right)}y^{\frac{\frac{b}{a}-1}{s}}}{n\,s\left(2-\frac{b}{a}\right)-\left(2-\frac{b}{a}\right)+1}-1\right)$
Uniform	$r+(s-r)y$	$\frac{1}{a(b-a)}\left(\frac{n^{2\left(2-\frac{b}{a}\right)}\{(s-r)y\}^{\frac{b}{a}-1}}{n\left(2-\frac{b}{a}\right)-\left(2-\frac{b}{a}\right)+1}-1\right)$

7. Conclusion

In this article, quantile based research of entropy measures found bigger interest among investigators as an alternate methodology of measuring uncertainty of a random variable. A very important non-additive generalization of Shannon entropy is that the Bilal's and Baig's entropy measure, the calculation of this measure is quite easy wherever the distribution functions are not tractable whereas the quantile functions have easier forms. we represent quantile version of this measure of order statistics for residual and past lifetimes and take a look at their properties. Furthermore, the uniform, exponential, generalized Pareto and finite range distributions, which are usually employed in the reliability modeling are characterized in terms of Bilal's and Baig's measure with extreme order statistics. The results obtained during this article are general in the sense that they reduce to some of the results for quantile based Shannon entropy and Methai-Haubold for order statistics once parameters approaches to unity.

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